Discussion Paper Series

Institute-Wide Joint Research Project
Methodology of Social Sciences: Measuring Phenomena and Values

Suzumura consistency and minimal quasi-transitive extensions

Walter Bossert
University of Montreal

Susumu Cato
University of Tokyo

Kohei Kamaga
Sophia University

September 12, 2023

E-23-006

Institute of Social Science, University of Tokyo
[https://web.iss.u-tokyo.ac.jp/methodology/en/dp/dp/]

Suzumura consistency and minimal quasi-transitive extensions*

Walter Bossert
Department of Economics
University of Montreal
P.O. Box 6128, Station Downtown
Montreal QC H3C 3J7
Canada
walter.bossert.wb@gmail.com

Susumu Cato
Institute of Social Science
University of Tokyo
7-3-1, Hongo, Bunkyo-ku
Tokyo 113-0033
Japan
susumu.cato@gmail.com

Kohei Kamaga
Faculty of Economics
Sophia University
7-1, Kioi-cho, Chiyoda-ku
Tokyo 102-8554
Japan
kohei.kamaga@sophia.ac.jp

This version: September 10, 2023

Abstract. Suzumura consistency and quasi-transitivity are important weakenings of transitivity. Suzumura consistency rules out goodness cycles with at least one instance of betterness, and quasi-transitivity requires betterness to be transitive. This paper examines to what extent these two (in general, independent) properties can be satisfied simultaneously. To do so, we define the concept of a minimal quasi-transitive extension for a Suzumura consistent relation. Because quasi-transitivity does not allow for the existence of a closure operator, this extension cannot be a closure. However, as we show, any Suzumura consistent relation can be extended to a quasi-transitive and Suzumura consistent relation. In addition, we apply the notion of a minimal quasi-transitive extension in the context of rational choice theory.

* Financial support from KAKENHI through grants Nos. JP18H05204, JP20H01446, JP20K01565, JP22K01387, JP22H05083, JP22H05086, and JP23H00814 is gratefully acknowledged.

1 Introduction

Although transitivity is considered a plausible coherence property of a binary relation, there are several reasons why it may be too demanding in some situations. Armstrong (1939) argues forcefully against the transitivity of equal goodness (which is implied by transitivity) because, inevitably, there are thresholds of perception that prevent this requirement from being acceptable in several circumstances. This is evidenced, for example, in Luce's (1956) well-known thought experiment that refers to a cup of coffee and the amount of sugar used. A cup that contains a certain amount of sugar is, for most people, indistinguishable from a cup that contains 0.1 grams more. If transitivity is applied repeatedly, a cup that contains 10 grams is indistinguishable from a cup that contains no sugar at all. This rather implausible implication is a direct consequence of assuming that equal goodness is transitive.

If a relation is not necessarily complete, another case can be made that transitivity may not be the most suitable coherence requirement. Transitivity requires that if an object x is at least as good as an object y and y is at least as good as z, then x must be at least as good as z. But if the underlying goodness relation is not complete, this conclusion does not appear to be all that compelling. If x is at least as good as y and y is at least as good as z, it is eminently reasonable to exclude the possibility that z be better that x—but there is nothing wrong with x and z being non-comparable.

There are (at least) two plausible candidates that involve relaxing the transitivity requirement. The first of these consists of weakening transitivity to Suzumura consistency (Suzumura, 1976), a property that excludes goodness cycles with at least one instance of betterness. In the presence of reflexivity and completeness, Suzumura consistency is equivalent to transitivity but this equivalence is not valid in general. As shown by Suzumura (1976), Suzumura consistency is necessary and sufficient for the existence of an ordering extension. This result constitutes a substantial strengthening of Szpilrajn's (1930) classical observation that transitivity is sufficient for this conclusion. In addition, Suzumura consistency allows for a well-defined closure operator. That any arbitrary relation R possesses a Suzumura consistent closure—that is, a smallest Suzumura consistent relation that contains R—is established in Bossert, Sprumont, and Suzumura (2005).

The second option is quasi-transitivity, a condition that has been studied extensively since Sen's (1969) seminal contribution. A goodness relation is quasi-transitive if its asymmetric (betterness) part is transitive; clearly, this does not imply that the symmetric (equal goodness) part be transitive, thus avoiding the threshold-of-perception issue pointed out by Armstrong (1939). This property is of importance because it guarantees path independent choice functions; see Plott (1973) and Blair, Bordes, Kelly, and Suzumura (1976). However, quasi-transitivity does not address the second problem alluded to above—it fails to rule out goodness cycles with at least one instance of betterness. Moreover, in contrast to Suzumura consistency, there is no such thing as a quasi-transitive closure because there is, in general, no unique way of obtaining a larger relation that removes violations of quasi-transitivity.

In this paper, we examine how Suzumura consistency and quasi-transitivity can be satisfied simultaneously. This is important because both of these properties possess some appealing attributes. As alluded to above, quasi-transitivity is a necessary and sufficient condition for path independent choice. In addition to the features already mentioned, Suzumura consistency is necessary and sufficient for the avoidance of a money pump (Raiffa, 1968, p. 78). This is the case because the property prevents an agent from engaging in a chain of exchanges each of which leaves him or her at least as well off as the object that is currently in his or her possession, only to arrive at a situation in which it is better to obtain the object he or she started out with. See Bossert and Suzumura (2010, pp. 4–5) for a discussion.

We begin by establishing the existence of what we refer to as a minimal quasi-transitive extension in the presence of Suzumura consistency. An extension of this nature does not exist in general and, therefore, this existence result reveals an important complementarity between Suzumura consistency and quasi-transitivity. We also show that the minimal quasi-transitive extension of a Suzumura consistent relation is Suzumura consistent. However, the minimal quasi-transitive extension still fails to be a closure even if Suzumura consistency is satisfied.

As an application, we focus on rational choice on arbitrary domains; see Richter (1966, 1971) for pioneering contributions. Characterizations of rationalizability by a Suzumura consistent relation or by a quasi-transitive relation are available in the existing literature; see, for example, Bossert, Sprumont, and Suzumura (2005) and Bossert and Suzumura (2009, 2010). Bossert and Suzumura (2012) examine the conjunction of Suzumura consistency and quasi-transitivity, with a focus on the ramifications for collective choice. However, to the best of our knowledge, the implications of combining Suzumura consistency and quasi-transitivity have not been explored in the context of rational choice. We begin by identifying some logical relationships if a rationalization is required to satisfy both Suzumura consistency and quasi-transitivity. A sufficient condition for Suzumura consistent and quasi-transitive rationalizability follows, with the notion of a minimal quasi-transitive extension emerging as its key element.

Section 2 introduces our basic notation and some relevant existing results concerning binary relations. Minimal quasi-transitive extensions are defined in Section 3, and we establish the importance of Suzumura consistency in this context. Section 4 examines rationalizing relations that are both Suzumura consistent and quasi-transitive. Section 5 concludes the paper.

2 Preliminaries

Suppose that X is a non-empty set of alternatives. Consider a (binary) relation $R \subseteq X \times X$ on X with asymmetric part $P(R)$, symmetric part $I(R)$, and non-comparable part $N(R)$. The diagonal relation on X is given by $\Delta = \{(x,x) \mid x \in X\}$. A relation R' is an extension of a relation R if $R \subseteq R'$ and $P(R) \subseteq P(R')$.

Two standard richness properties of a relation are those of reflexivity and completeness. A relation R is reflexive if $(x,x) \in R$ for all $x \in X$, and R is complete if $(x,y) \in R$ or $(y,x) \in R$ for all $x,y \in X$ such that $x \neq y$. These two axioms are frequently combined into a single condition but, for our purposes, it is of importance to state them as separate properties. One motivation for doing so is that reflexivity is considerably more uncontroversial than completeness. Moreover, as is the case for some observations reported in Bossert and Suzumura (2010), reflexivity cannot be taken for granted and, therefore, distinguishing the two can be of crucial importance in some applications.

There is another class of conditions the members of which require relations to exhibit some form of coherence. The most prominent of these is transitivity. A relation R is transitive if the conjunction of $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all $x, y, z \in X$.

The transitive closure $tc(R)$ of R is the unique smallest transitive relation that contains R. It is defined by

$$\begin{aligned} tc(R) \;=\; & \{(x, y) \in X \times X \mid \exists K \in \mathbb{N} \text{ and } x^0, \ldots, x^K \in X \text{ such that} \\ & x = x^0 \text{ and } (x^{k-1}, x^k) \in R \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y\}. \end{aligned}$$

The transitive closure $tc(R)$ possesses the properties of a closure operator; the corresponding result is stated as Theorem 2.1 of Bossert and Suzumura (2010).

Theorem 1. *Suppose that R and R' are relations on X.*

(i) $R \subseteq tc(R)$.

(ii) R *is transitive if and only if* $R = tc(R)$.

(iii) $tc(R) \subseteq Q$ *for all transitive relations* Q *such that* $R \subseteq Q$.

(iv) If $R \subseteq R'$, *then* $tc(R) \subseteq tc(R')$.

A reflexive and transitive relation is called a quasi-ordering, and we refer to a complete quasi-ordering as an ordering. If an extension R' of R is an ordering, R' is called an ordering extension. As shown by Arrow (1951; 1963) and Hansson (1968), any quasi-ordering has an ordering extension; see Szpilrajn (1930) for the original extension theorem expressed in terms of a transitive and irreflexive relation. (A relation R is irreflexive if $(x, x) \notin R$ for all $x \in X$.) The Szpilrajn-Arrow-Hansson theorem establishes that transitivity is a sufficient condition for the existence of an ordering extension. This result is generalized by Suzumura (1976) who weakens transitivity to a condition that is both necessary and sufficient. Suzumura (1976) introduces the property as consistency but we refer to it as Suzumura consistency to distinguish it from other (unrelated) forms of consistency conditions that appear in the literature.

A relation R is Suzumura consistent if $(x, y) \in tc(R)$ implies $(y, x) \notin P(R)$ for all $x, y \in X$. Transitivity implies Suzumura consistency but the reverse implication is not valid in general. If R is reflexive and complete, Suzumura consistency and transitivity are equivalent. As alluded to above, Suzumura consistency is a necessary and sufficient condition for the existence of an ordering extension; see Suzumura (1976, 1983) and Bossert and Suzumura (2010) for proofs of this observation.

The Suzumura consistent closure $sc(R)$ of R is the unique smallest Suzumura consistent relation that contains R; this closure operator is introduced by Bossert, Sprumont, and Suzumura (2005). It is given by

$$sc(R) = R \cup \{(x, y) \in X \times X \mid (x, y) \in tc(R) \text{ and } (y, x) \in R\}.$$

In analogy to Theorem 1, we obtain the following result which reproduces Theorem 2.4 of Bossert and Suzumura (2010).

Theorem 2. *Suppose that R and R' are relations on X.*

(i) $R \subseteq sc(R) \subseteq tc(R)$.

(ii) R is Suzumura consistent if and only if $R = sc(R)$.

(iii) $sc(R) \subseteq Q$ *for all Suzumura consistent relations Q such that* $R \subseteq Q$.

(iv) If $R \subseteq R'$, *then* $sc(R) \subseteq sc(R')$.

An alternative weakening of transitivity is obtained by requiring the asymmetric part $P(R)$ of R to be transitive. Thus, R is quasi-transitive if the conjunction of $(x, y) \in P(R)$ and $(y, z) \in P(R)$ implies $(x, z) \in P(R)$ for all $x, y, z \in X$. We note that R is quasi-transitive if and only if $P(R) = tc(P(R))$. Transitivity implies quasi-transitivity and, without further assumptions, the properties of Suzumura consistency and quasi-transitivity are independent. In the presence of reflexivity and completeness, Suzumura consistency implies quasi-transitivity as an immediate consequence of the former's equivalence to transitivity in this case.

Although quasi-transitivity has attracted considerable attention as a weakening of transitivity (see, for example, Sen, 1969, 1970), the property suffers from the shortcoming that it does not possess a well-defined closure operator. For example, suppose that $X = \{x, y, z\}$, and consider the relation

$$R = \{(x, y), (y, z), (x, z), (z, x)\}. \tag{1}$$

By definition, $P(R) = \{(x, y), (y, z)\}$ and $I(R) = \{(x, z), (z, x)\}$. It is immediate that both of $R' = R \cup \{(y, x)\}$ and $R'' = R \cup \{(z, y)\}$ are quasi-transitive relations that contain R but there is no smallest relation that possesses this property. This implies that a quasi-transitive closure cannot be defined.

The final coherence property we introduce is that of acyclicity, which rules out the existence of cycles that are composed of asymmetric relationships. A relation R is acyclical if $(x, y) \in tc(P(R))$ implies $(y, x) \notin P(R)$ for all $x, y \in X$. It is immediate that Suzumura consistency implies acyclicity, and so does quasi-transitivity. The reverse implications are not valid.

3 Minimal quasi-transitive extensions

The existence of a quasi-transitive extension is not guaranteed in general. For example, consider again the set $X = \{x, y, z\}$ and the relation R defined in (1). To see that this relation does not have a quasi-transitive extension, observe that any quasi-transitive extension R' of R must be such that $(x, z) \in P(R')$. Thus, $(z, x) \notin R'$ so that R' cannot contain R because $(z, x) \in R$ by definition. This example involves a relation R that fails to be Suzumura consistent. By Suzumura's (1976) extension result, any Suzumura consistent relation has an ordering extension. Because an ordering is quasi-transitive, it follows immediately that Suzumura consistency is a sufficient condition for the existence of a quasi-transitive extension.

A more subtle question is whether a relation possesses what we refer to as a minimal quasi-transitive extension. In analogy to the notion of a closure operator, an extension R' of a relation R with a given property is minimal if all extensions with that property contain R'. Thus, a transitive extension R' of a relation R is minimal if $R' \subseteq Q$ for all transitive extensions Q of R, and a quasi-transitive extension R' of a relation R is minimal if $R' \subseteq Q$ for all quasi-transitive extensions Q of R. Clearly, an extension that is minimal in this sense must be unique. If a relation R is Suzumura consistent, its minimal transitive extension is well-defined and given by the transitive closure $tc(R)$. This follows from the observation that $R \subseteq R'$ for any transitive extension R' of R and Theorem 1, which implies that $tc(R) \subseteq tc(R') = R'$.

However, the transitive closure $tc(R)$ of a Suzumura consistent relation R does not have to be the minimal quasi-transitive extension. For example, suppose that $X = \{x, y, z\}$ and $R = \{(x, y), (y, z), (z, y)\}$. It follows that $P(R) = \{(x, y)\}$ and $I(R) = \{(y, z), (z, y)\}$. This relation is quasi-transitive and Suzumura consistent. Because R is quasi-transitive, it trivially is a quasi-transitive extension of itself. But we also have $tc(R) = R \cup \{(x, z)\}$, which implies that $tc(R)$ cannot be the minimal quasi-transitive extension of R because $tc(R)$ contains the pair (x, z) and R does not.

As we prove later in this section, the minimal quasi-transitive extension $qe(R)$ of a Suzumura consistent relation R is well-defined and given by

$$qe(R) = R \cup tc(P(R)).$$

Our first observation is that, for any relation R on X, the transitive closure $tc(R)$ of R and the transitive closure $tc(qe(R))$ of the relation $qe(R)$ coincide.

Theorem 3. *For any relation R on X, $tc(R) = tc(qe(R))$.*

Proof. By definition of $qe(R)$, $R \subseteq qe(R)$. Thus, it follows from Theorem 1 that $tc(R) \subseteq tc(qe(R))$. To show that $tc(qe(R)) \subseteq tc(R)$, let $(x, y) \in tc(qe(R))$. By definition of $tc(qe(R))$, there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x = x^0$, $x^K = y$, and $(x^{k-1}, x^k) \in qe(R)$ for all $k \in \{1, \ldots, K\}$. By definition of $qe(R)$, for all $k \in \{1, \ldots, K\}$, $(x^{k-1}, x^k) \in R$ or $(x^{k-1}, x^k) \in tc(P(R))$. If $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$, then $(x, y) \in tc(R)$. Now define

$$\mathcal{K} = \{k \in \{1, \ldots, K\} \mid (x^{k-1}, x^k) \in tc(P(R))\} \neq \emptyset$$

and let $k \in \mathcal{K}$. By definition of $tc(P(R))$, there exist $L_k \in \mathbb{N}$ and $x^{k,0}, \ldots, x^{k,L_k} \in X$ such that $x^k = x^{k,0}$, $x^{k,L_k} = x^{k+1}$, and $(x^{k,\ell-1}, x^{k,\ell}) \in P(R) \subseteq R$ for all $\ell \in \{1, \ldots, L_k\}$. Thus, letting $K^* = K - |\mathcal{K}| + \sum_{k \in \mathcal{K}} L_k$, there exist $x^0, \ldots, x^{K^*}$ such that $x = x^0$, $x^{K^*} = y$, and $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K^*\}$. By definition of $tc(R)$, $(x, y) \in tc(R)$. ■

We now derive the asymmetric and symmetric parts of $qe(R)$, provided that R is Suzumura consistent.

Theorem 4. *Suppose that R is a Suzumura consistent relation on X. The asymmetric part $P(qe(R))$ of $qe(R)$ is given by*

$$P(qe(R)) = tc(P(R)). \tag{2}$$

Proof. To prove that (2) is true, suppose that R is Suzumura consistent. Let $x, y \in X$. By definition of $qe(R)$, it follows that $(x, y) \in P(qe(R))$ if and only if

$$[(x, y) \in R \text{ or } (x, y) \in tc(P(R))] \text{ and } (y, x) \notin R \text{ and } (y, x) \notin tc(P(R))$$

which is equivalent to the disjunction

$$(x, y) \in R \text{ and } (y, x) \notin R \text{ and } (y, x) \notin tc(P(R)) \tag{3}$$

or

$$(x, y) \in tc(P(R)) \text{ and } (y, x) \notin R \text{ and } (y, x) \notin tc(P(R)). \tag{4}$$

Observe first that (3) is equivalent to

$$(x, y) \in R \text{ and } (y, x) \notin R$$

because $(y, x) \notin tc(P(R))$ is implied by $(x, y) \in R$ and the Suzumura consistency of R; note that the conjunction of $(x, y) \in R$ and $(y, x) \notin tc(P(R))$ would lead to a cycle with at least one instance of an asymmetric relationship between two consecutive alternatives, in contradiction to Suzumura consistency.

Analogously, (4) is equivalent to

$$(x, y) \in tc(P(R)) \text{ and } (y, x) \notin tc(P(R))$$

because the conjunction of $(x, y) \in tc(P(R))$ and $(y, x) \in R$ would, in violation of Suzumura consistency, lead to a cycle with at least one instance of an asymmetric relationship between two consecutive alternatives. Using the definition of the asymmetric part of a relation, we obtain

$$P(qe(R)) = P(R) \cup P(tc(P(R))).$$

Furthermore, $tc(P(R))$ is asymmetric; otherwise, there exist $x, y \in X$ such that $(x, y) \in tc(P(R))$ and $(y, x) \in tc(P(R))$, which is a contradiction because R is Suzumura consistent. Thus, $P(tc(P(R))) = tc(P(R))$. From Theorem 1, $P(R) \subseteq tc(P(R))$ and, therefore, (2) follows. ■

If R is not Suzumura consistent, (2) does not follow. Consider, again, the set $X = \{x, y, z\}$ and the relation defined by (1). We obtain $qe(R) = R$ and hence $P(qe(R)) = P(R) = \{(x, y), (y, z)\}$. However, because $tc(P(R)) = P(R) \cup \{(x, z)\}$, (2) is not implied.

Theorem 5. *Suppose that R is a Suzumura consistent relation on X. The symmetric part $I(qe(R))$ of $qe(R)$ is given by*

$$I(qe(R)) = I(R). \tag{5}$$

Proof. Suppose that R is Suzumura consistent. By definition of $qe(R)$, $I(R) \subseteq I(qe(R))$. To show that $I(qe(R)) \subseteq I(R)$, let $(x, y) \in I(qe(R))$. Then, $(x, y) \notin tc(P(R))$ because, otherwise, we obtain $(x, y) \in P(qe(R))$ by Theorem 4, a contradiction. By the same argument, $(y, x) \notin tc(P(R))$. Thus, $(x, y) \in R$ and $(y, x) \in R$, that is, $(x, y) \in I(R)$. ■

Again, (5) does not follow if the assumption of Suzumura consistency is dropped. Let $X = \{x, y, z\}$ and $R = \{(x, y), (y, z), (z, x)\}$. We obtain $I(R) = \emptyset$ and $qe(R) = R \cup \{(y, x), (z, y), (x, z)\}$. Therefore,

$$I(qe(R)) = qe(R) \neq \emptyset = I(R).$$

Our next observation is that if R is a Suzumura consistent relation, then the relation $qe(R)$ is Suzumura consistent as well.

Theorem 6. *Suppose that R is a Suzumura consistent relation on X. The relation $qe(R)$ is Suzumura consistent.*

Proof. To prove the contrapositive statement, suppose that $qe(R)$ is not Suzumura consistent. Then there exist $x, y \in X$ such that $(x, y) \in tc(qe(R))$ and $(y, x) \in P(qe(R))$. By Theorem 3, $(x, y) \in tc(R)$. Furthermore, by Theorem 4, $(y, x) \in tc(P(R))$. By definition of Suzumura consistency, the conjunction of $(y, x) \in tc(P(R))$ and $(x, y) \in tc(R)$ implies that R is not Suzumura consistent. ■

The main result of this section establishes that $qe(R)$ is the minimal quasi-transitive extension of a Suzumura consistent relation R.

Theorem 7. *Suppose that R is a Suzumura consistent relation on X. The minimal quasi-transitive extension of R is given by $qe(R)$.*

Proof. Suppose that R is a Suzumura consistent relation on X. We first prove that $qe(R)$ is quasi-transitive. Let $x, y, z \in X$. Using (2), it follows that $(x, y) \in P(qe(R))$ and $(y, z) \in P(qe(R))$ if and only if

$$(x, y) \in tc(P(R)) \text{ and } (y, z) \in tc(P(R))$$

which implies $(x, z) \in tc(P(R))$. Because R is Suzumura consistent, we obtain $(x, z) \in P(qe(R))$ by (2). Therefore, $qe(R)$ is quasi-transitive.

Clearly, R is a subset of $qe(R) = R \cup tc(P(R))$. Using (2), it follows from Theorem 1 that $P(R) \subseteq tc(P(R)) = P(qe(R))$ so that $qe(R)$ is an extension of R if R is Suzumura consistent.

We complete the proof by showing that $qe(R)$ is the minimal quasi-transitive extension of R. Suppose that Q is a quasi-transitive extension of R. Let $x, y \in X$, and suppose that $(x, y) \in qe(R)$. By definition,

$$(x, y) \in R \text{ or } (x, y) \in tc(P(R)).$$

If $(x, y) \in R$, $(x, y) \in Q$ follows immediately because $R \subseteq Q$ by definition of an extension.

If $(x, y) \in tc(P(R))$, it follows that there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$, and $x^K = y$. Because Q is an extension of R, it follows that $P(R) \subseteq P(Q)$ and, therefore, $(x^{k-1}, x^k) \in P(Q)$ for all $k \in \{1, \ldots, K\}$. Because Q is quasi-transitive, we obtain $(x, y) \in P(Q)$, as desired. ■

A quasi-transitive closure need not exist even in the presence of Suzumura consistency. For example, suppose that $X = \{x, y, z\}$ and $R = \{(x, y), (y, z)\}$. This relation is Suzumura consistent and, by Theorem 7, $qe(R) = R \cup \{(x, z)\}$ is its minimal quasi-transitive extension. However, this relation is not the smallest quasi-transitive relation containing the original relation R. Indeed, $R' = R \cup \{(y, x)\}$ is a quasi-transitive relation that contains R but R' is not a subset of $qe(R)$.

The observations of the following theorem parallel those of Theorems 1 and 2.

Theorem 8. *Suppose that R is a Suzumura consistent relation on X.*

(i) $R \subseteq qe(R)$.

(ii) R is quasi-transitive if and only if $R = qe(R)$.

(iii) $qe(R) \subseteq Q$ for all quasi-transitive extensions Q of R.

(iv) There exist Suzumura consistent relations R and R' such that $R \subseteq R'$ and $qe(R) \not\subseteq qe(R')$.

Proof. Suppose that R is a Suzumura consistent relation on X.

(i) By definition, $R \subseteq R \cup tc(P(R)) = qe(R)$.

(ii) If R is quasi-transitive, it follows that $tc(P(R)) \subseteq R$ and hence $qe(R) = R$. Conversely, if $R = qe(R)$, R is quasi-transitive because $qe(R)$ is.

(iii) This part is the result of Theorem 7.

(iv) An example is sufficient to prove this part. Let $X = \{x, y, z\}$, $R = \{(x, y), (y, z)\}$, and $R' = \{(x, y), (y, z), (z, y)\}$. It follows that $qe(R) = R \cup \{(x, z)\}$ and $qe(R') = R'$. Thus, $R \subseteq R'$ and $qe(R) \not\subseteq qe(R')$ because $(x, z) \in qe(R)$ and $(x, z) \notin qe(R')$. ■

Part (i) of this theorem is valid even if R is not Suzumura consistent; we stated this requirement on R in the preamble of the theorem statement to simplify the exposition.

Part (iii) of Theorem 8 merely establishes a minimality property within the class of quasi-transitive extensions, whereas the corresponding conclusions in the two earlier theorems apply to all relations that contain R. This is the case because Theorems 1 and 2 list the properties of the requisite closure operator rather than those of an extension.

Most significantly, part (iv) of Theorem 8 shows that the operator qe violates a fundamental property of a closure operator.

We conclude this section with an observation regarding the relationship between $tc(R)$ and $qe(R)$.

Theorem 9. *Suppose that R is a Suzumura consistent relation on X. The transitive closure $tc(R)$ of R is an extension of the minimal quasi-transitive extension $qe(R)$ of R.*

Proof. Suppose that R is Suzumura consistent. By Theorem 6, $qe(R)$ is Suzumura consistent. By Lemma 3 of Cato (2012), the Suzumura consistency of a relation R' implies that $tc(R')$ is an extension of R'. Thus, $tc(qe(R))$ is an extension of $qe(R)$. By Theorem 3, $tc(R) = tc(qe(R))$. Therefore, $tc(R)$ is an extension of $qe(R)$. ■

4 Rational choice

A fundamental question in decision theory is whether observed (or observable) choices can be rationalized in the sense that they are performed with a purpose in mind. The notion of purposive choice is typically captured in terms of a rationalizing relation according to which the observed choices are the greatest elements or the maximal elements according to this relation.

The analysis of rational choice has a long tradition in economic theory. In the context of consumer demand, Samuelson's (1938a,b) seminal contributions examined whether observed demand functions can be interpreted as resulting from an optimization problem solved by the consumer in question. Specifically, the fundamental objective of revealed preference theory as developed by Samuelson is to define testable restrictions on observed demand functions that allow us to conclude whether these choices can be generated from solving the problem of choosing the best bundles from those that are feasible according to an underlying relation defined on the commodity space.

In the special case pioneered by Samuelson, the feasible set of options in a given situation consists of all possible bundles that are affordable with the consumer's budget at the current prices. Richter (1966, 1971) adopts a considerably more general approach to rational choice by allowing a choice function to be defined on any arbitrary non-empty domain, and this is the basic framework that we employ in this section.

Let $\mathcal{X}$ denote the set of all non-empty subsets of X, and suppose that $\Sigma \subseteq \mathcal{X}$ is a non-empty domain. A choice function is a mapping $C\colon \Sigma \to \mathcal{X}$ such that, for all $S \in \Sigma$, $C(S) \subseteq S$. The direct revealed goodness relation R_C associated with a choice function C is defined by

$$R_C = \{(x, y) \in X \times X \mid \exists S \in \Sigma \text{ such that } x \in C(S) \text{ and } y \in S\}.$$

For a feasible set $S \in \Sigma$ and a relation R on X, the set $G(S, R)$ of greatest elements in S according to R is

$$G(S, R) = \{x \in S \mid (x, y) \in R \text{ for all } y \in S\}.$$

Analogously, the set $M(S, R)$ of maximal elements in $S \in \Sigma$ according to a relation R on X is

$$M(S, R) = \{x \in S \mid (y, x) \notin P(R) \text{ for all } y \in S\}.$$

By definition, $G(S, R) \subseteq M(S, R)$ for all $S \in \Sigma$ and for all relations R on X. Furthermore, $G(S, R) = M(S, R)$ if R is reflexive and complete. See, for example, Bossert and Suzumura (2010, Theorem 2.5) for proofs of these two observations.

A choice function C is greatest-element rationalizable if there exists a relation R on X such that $C(S) = G(S, R)$ for all $S \in \Sigma$. In this case, we say that R greatest-element rationalizes C and we refer to R as a greatest-element rationalization of C. Analogously, C is maximal-element rationalizable if there exists a relation R on X such that $C(S) = M(S, R)$ for all $S \in \Sigma$. In this case, R maximal-element rationalizes C and R is a maximal-element rationalization of C.

In the traditional setup pioneered by Samuelson, the notion of rationalizability was linked to the assumption that a rationalizing relation be an ordering. However, rationalizability can be studied without any commitment to additional richness properties such as reflexivity and completeness or coherence requirements such as transitivity; see, in particular, Richter (1971) for an early approach that examines more general notions of rationality.

In line with the subject matter of this paper, we consider definitions of rationalizability such that rationalizing relations are required to possess both coherence properties of quasi-transitivity and Suzumura consistency. This yields eight new notions of rationalizability. For each of the two fundamental types of rationalizability (greatest-element rationalizability and maximal-element rationalizability), there are four potentially distinct definitions. These correspond to the possibilities of (i) requiring none of the two richness properties of reflexivity and completeness; (ii) requiring reflexivity only; (iii) requiring completeness only; and (iv) requiring both reflexivity and completeness. To the best of our knowledge, these notions of rationalizability have not been explicitly examined in the earlier literature. A full account of the logical relationships among all 40 definitions of greatest-element rationalizability and maximal-element rationalizability that can be obtained by requiring none, one, or both of the two richness properties, and none or one of the four coherence properties of transitivity, Suzumura consistency, quasi-transitivity, and acyclicity can be found in Bossert and Suzumura (2009, 2010). As shown in these contributions, several of the properties are equivalent and there remain 11 distinct versions of greatest-element rationalizability and four versions of maximal-element rationalizability. In addition, each of the four distinct versions of maximal-element rationalizability is equivalent to one of the 11 versions of greatest-element rationalizability so that maximal-element rationalizability can be considered redundant; see Bossert and Suzumura (2010, Chapter 3) for details.

The following three theorems establish the logical relationships between the notions of rationalizability that can be defined if both quasi-transitivity and Suzumura consistency are to be satisfied by a rationalizing relation. We begin with greatest-element rationalizability.

Theorem 10. *Suppose that* $C\colon \Sigma \to \mathcal{X}$ *is a choice function.*

(i) C *is greatest-element rationalizable by a reflexive, complete, Suzumura consistent, and quasi-transitive relation if and only if* C *is greatest-element rationalizable by a complete, Suzumura consistent, and quasi-transitive relation.*

(ii) If C *is greatest-element rationalizable by a complete, Suzumura consistent, and quasi-transitive relation, then* C *is greatest-element rationalizable by a reflexive, Suzumura consistent, and quasi-transitive relation. The reverse implication is not valid.*

(iii) If C is greatest-element rationalizable by a reflexive, Suzumura consistent, and quasi-transitive relation, then C is greatest-element rationalizable by a Suzumura consistent and quasi-transitive relation. The reverse implication is not valid.

Proof. Let $C\colon \Sigma \to \mathcal{X}$ be a choice function.

(i) That greatest-element rationalizability by a reflexive, complete, Suzumura consistent, and quasi-transitive relation implies greatest-element rationalizability by a complete, Suzumura consistent, and quasi-transitive relation is immediate. To prove the reverse implication, suppose that R is a complete, Suzumura consistent, and quasi-transitive greatest-element rationalization of C. Define

$$R' = R \cup \Delta \cup \{(x,y) \mid x \in C(\Sigma) \text{ and } y \notin C(\Sigma)\} \setminus \{(y,x) \mid x \in C(\Sigma) \text{ and } y \notin C(\Sigma)\}.$$

As shown in part (a) of the proof of Theorem 3.2 in Bossert and Suzumura (2010), R' is a reflexive, complete, and Suzumura consistent rationalization of C. It remains to be shown that R' is quasi-transitive. Suppose that $(x,y),(y,z) \in P(R')$ for three alternatives $x,y,z \in X$. Clearly, x, y, and z must be pairwise distinct. If $(z,x) \in R'$, we obtain a contradiction to the Suzumura consistency of R'. Because R' is complete, it follows that $(x,z) \in P(R')$ so that R' is quasi-transitive.

(ii) The implication follows immediately from part (i). To prove that the reverse implication is not valid, consider the following example; see also parts (i) and (l) of the proof of Theorem 3.2 in Bossert and Suzumura (2010). Let $X = \{x,y,z\}$,

$$\Sigma = \{\{x\},\{x,y\},\{x,y,z\},\{x,z\}\},$$

$C(\{x\}) = \{x\}$, $C(\{x,y\}) = \{x,y\}$, $C(\{x,y,z\}) = \{x\}$, and $C(\{x,z\}) = \{x,z\}$. The choice function C is greatest-element rationalized by the reflexive, Suzumura consistent, and quasi-transitive relation

$$R = \{(x,x),(x,y),(x,z),(y,x),(y,y),(z,x),(z,z)\}.$$

Suppose that R' is a complete rationalization of C. By the completeness of R', it follows that $(y,z) \in R'$ or $(z,y) \in R'$. If $(y,z) \in R'$, it follows that $(y,x) \in R'$ and $(y,y) \in R'$ because R' is a greatest-element rationalization of C and $y \in C(\{x,y\})$. Together with $(y,z) \in R'$ and the definition of greatest-element rationalizability, we obtain $y \in C(X)$, contradicting the definition of C. Analogously, if $(z,y) \in R'$, it follows that $(z,x) \in R'$ and $(z,z) \in R'$. Together with $(z,y) \in R'$ and the definition of greatest-element rationalizability, we obtain $z \in C(X)$, again contradicting the definition of C.

(iii) The implication of the statement is immediate. To show that the reverse implication is not valid, consider the following example. Let $X = \{x,y,z,v\}$,

$$\Sigma = \{\{x,y\},\{x,y,z,v\},\{y,z\},\{y,z,v\}\},$$

$C(\{x,y\}) = \{y\}$, $C(\{x,y,z,v\}) = \{v\}$, $C(\{y,z\}) = \{z\}$, and $C(\{y,z,v\}) = \{z,v\}$. This choice function is greatest-element rationalized by the Suzumura consistent and quasi-transitive relation

$$R = \{(x,y),(y,x),(y,y),(z,y),(z,z),(z,v),(v,x),(v,y),(v,z),(v,v)\}.$$

Suppose that R' is a reflexive, Suzumura consistent, and quasi-transitive rationalization of C. By reflexivity, $(x, x) \in R'$ and, because $x \notin C(\{x, y\})$ and $y \in C(\{x, y\})$, we have $(y, x) \in P(R')$. Similarly, because $y \notin C(\{y, z\})$ and $z \in C(\{y, z\})$, we obtain $(z, y) \in P(R')$. Since R' is quasi-transitive, it follows that $(z, x) \in P(R')$. Because $z \in C(\{y, z, v\})$, we obtain $(z, y) \in P(R') \subseteq R'$, $(z, z) \in R'$, and $(z, v) \in R'$. Because R' is a greatest-element rationalization of C and $(z, x) \in P(R') \subseteq R'$, we must have $z \in C(\{x, y, z, v\})$, which contradicts the definition of C. ■

Because Suzumura consistency and transitivity are equivalent in the presence of reflexivity and completeness, and quasi-transitivity is implied by transitivity, the two definitions that appear in part (i) of Theorem 10 are equivalent to greatest-element rationalizability by an ordering—the strongest form of rationalizability that can be obtained by employing the richness properties of reflexivity and completeness, along with none or one of the four coherence properties. Therefore, introducing greatest-element rationalizability by a (reflexive,) complete, Suzumura consistent, and quasi-transitive relation does not yield any new independent properties that are not already covered by known variants. Greatest-element rationalizability by a reflexive, Suzumura consistent, and quasi-transitive relation, on the other hand, is not equivalent to any of the existing versions, and the same is true for the property that results if the reflexivity requirement is dropped. The implications that have to be added in the system of logical relationships are that greatest-element rationalizability by a reflexive, Suzumura consistent, and quasi-transitive relation implies greatest-element rationalizability by a reflexive and quasi-transitive relation, and that greatest-element rationalizability by a Suzumura consistent and quasi-transitive relation implies greatest-element rationalizability by a quasi-transitive relation.

The following theorem exhibits the logical relationships among the four notions of maximal-element rationalizability that can be defined if rationalizing relations are required to be Suzumura consistent and quasi-transitive.

Theorem 11. *Suppose that $C\colon \Sigma \to \mathcal{X}$ is a choice function.*

(i) C is maximal-element rationalizable by a reflexive, complete, Suzumura consistent, and quasi-transitive relation if and only if C is maximal-element rationalizable by a complete, Suzumura consistent, and quasi-transitive relation.

(ii) C is maximal-element rationalizable by a reflexive, Suzumura consistent, and quasi-transitive relation if and only if C is maximal-element rationalizable by a Suzumura consistent and quasi-transitive relation.

(iii) If C is maximal-element rationalizable by a complete, Suzumura consistent, and quasi-transitive relation, then C is maximal-element rationalizable by a Suzumura consistent and quasi-transitive relation. The reverse implication is not valid.

Proof. Let $C\colon \Sigma \to \mathcal{X}$ be a choice function.

(i) That maximal-element rationalizability by a reflexive, complete, Suzumura consistent, and quasi-transitive relation implies maximal-element rationalizability by a complete, Suzumura consistent, and quasi-transitive relation is immediate. To prove the reverse implication, suppose that R is a complete, Suzumura consistent, and quasi-transitive maximal-element rationalization of C. Let $R' = R \cup \Delta$. Clearly, R' is reflexive. Moreover, R' is complete, Suzumura consistent, and quasi-transitive because these properties are unaffected if the pairs in the diagonal Δ are added to R. Analogously, R' maximal-element rationalizes C because R does and the addition of the pairs in the diagonal leaves this property unchanged because only the asymmetric part of a maximal-element rationalization is relevant.

(ii) The proof of this part parallels that of part (i); the completeness of R and R' (or the lack thereof) does not change the argument employed.

(iii) The implication of the statement is immediate. To show that the reverse implication is not valid, consider the following example; see also part (e) of the proof of Theorem 3.3 in Bossert and Suzumura (2010). Let $X = \{x, y, z\}$,

$$\Sigma = \{\{x, y\}, \{x, z\}, \{y, z\}\},$$

$C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{z\}$, and $C(\{y, z\}) = \{y, z\}$. This choice function is maximal-element rationalized by the Suzumura consistent and quasi-transitive relation

$$R = \{(z, x)\}.$$

Suppose that R' is a complete, Suzumura consistent, and quasi-transitive rationalization of C. By the definition of maximal-element rationalizability, we must have $(z, x) \in P(R')$ because $x \notin C(\{x, z\})$. Because R' is complete and maximal-element rationalizes C, it follows that $(x, y) \in I(R')$ and $(y, z) \in I(R')$. This is a contradiction to the Suzumura consistency of R'. ■

Because Suzumura consistency and transitivity are equivalent in the presence of reflexivity and completeness, and quasi-transitivity is implied by transitivity, it follows that maximal-element rationalizability by a reflexive, complete, Suzumura consistent, and quasi-transitive relation is equivalent to maximal-element rationalizability by a reflexive, complete, and transitive relation. Therefore, the two definitions of maximal-element rationalizability that appear in part (i) of Theorem 11 do not add any new independent versions of rationality; they are covered by some of the known variants.

Moreover, maximal-element rationalizability by a Suzumura consistent and quasi-transitive relation is equivalent to maximal-element rationalizability by a quasi-transitive relation. To see that maximal-element rationalizability by a quasi-transitive relation implies maximal-element rationalizability by a Suzumura consistent and quasi-transitive relation, suppose that R is a quasi-transitive rationalization of a choice function C. Because quasi-transitivity implies acyclicity, R is acyclical. Define $R' = R \setminus I(R)$. Therefore, $P(R') = P(R)$ and $I(R') = \emptyset$. It follows that R' is a quasi-transitive maximal-element rationalization of C because only the asymmetric part of a relation matters for the property of quasi-transitivity and maximal-element rationalizability. Because quasi-transitivity implies acyclicity, R' is also Suzumura consistent because acyclicity and Suzumura consistency are equivalent as a consequence of the emptiness of $I(R')$. Conversely, maximal-element rationalizability by a Suzumura consistent and quasi-transitive relation trivially implies rationalizability by a quasi-transitive relation. Again, we conclude that no new independent variants of maximal-element rationalizability emerge if both Suzumura consistency and quasi-transitivity are required.

Necessary and sufficient conditions for those definitions of greatest-element rationalizability that involve quasi-transitive or acyclical greatest-element rationalizations are difficult to come by, and the same is true for complete greatest-element rationalizations without any coherence properties. That this is the case is explained in Bossert and Suzumura (2010, Chapter 4). Although Bossert and Suzumura provide such sets of conditions for all earlier forms of greatest-element rationalizability, those that involve transitive or Suzumura consistent greatest-element rationalizations (or greatest-element rationalizations that need not be complete and need not possess any coherence properties) are based on considerably more intuitive axioms than the remaining variants. The principal reason why transitive and Suzumura consistent notions of greatest-element rationalizability are more tractable than those that involve quasi-transitivity or acyclicity is that they allow for a well-defined closure operator.

One of the fundamental insights that already appears in Samuelson's (1938a,b) seminal work in the context of consumer choice is that a greatest-element rationalization R must respect the direct revealed goodness relation of a choice function in the sense that R_C is contained in R—that is, the subset relationship $R_C \subseteq R$ must be true for any greatest-element rationalization R of a choice function C. This observation is not difficult to prove. If $(x, y) \in R_C$ and R greatest-element rationalizes C, there is a feasible set $S \in \Sigma$ such that $x \in C(S)$ and $y \in S$. That $(x, y) \in R$ must be true now follows immediately from the definition of greatest-element rationalizability. As demonstrated by Richter (1971), this reasoning extends to transitive greatest-element rationalizability: if a transitive relation R greatest-element rationalizes a choice function C, it must be the case that $tc(R_C) \subseteq R$—otherwise we immediately obtain a contradiction to the definition of greatest-element rationalizability or to transitivity because the transitive closure $tc(R_C)$ of R_C is the smallest transitive relation that contains R_C. Bossert, Sprumont, and Suzumura (2005) apply the same argument to Suzumura consistent greatest-element rationalizability. Again, because the Suzumura consistent closure $sc(R_C)$ is the smallest Suzumura consistent relation that contains R_C, any Suzumura consistent greatest-element rationalization R must respect this relation so that the subset relationship $sc(R_C) \subseteq R$ follows immediately as a necessary condition.

These observations lead to intuitively appealing compatibility requirements that are necessary and sufficient for the three types of greatest-element rationalizability just discussed. A necessary and sufficient condition for the greatest-element rationalizability of a choice function C by an arbitrary relation is obtained by requiring that, for all feasible sets $S \in \Sigma$ and for all alternatives $x \in S$, $(x, y) \in R_C$ for all $y \in S$ implies $x \in C(S)$. That this axiom is indeed necessary and sufficient for rational choice can be proven using elementary methods; see Richter (1971). If R_C is replaced with $tc(R_C)$ in this requirement, a necessary and sufficient condition for transitive greatest-element rationalizability results; again, see Richter (1971) for a simple and intuitive proof. Finally, as demonstrated by Bossert, Sprumont, and Suzumura (2005), using the Suzumura consistent closure $sc(R_C)$ in place of R_C or $tc(R_C)$ leads to a necessary and sufficient condition for greatest-element rationalizability by a Suzumura consistent relation.

Because there is no such thing as a closure operator for quasi-transitivity and acyclicity (and, in fact, completeness), conditions of the type defined above cannot be formulated in these cases. For this reason, the sets of necessary and sufficient conditions for the remaining definitions of greatest-element rationalizability are more cumbersome—and they typically involve existential clauses because the uniqueness that comes with a well-defined closure operator is not present. The two definitions of greatest-element rationalizability by a Suzumura consistent and quasi-transitive relation without completeness suffer from the same fate, for the same reason. However, we can utilize the minimal quasi-transitive extension of a Suzumura consistent relation introduced in the previous section to at least obtain a sufficient condition for greatest-element rationalizability by a reflexive, Suzumura consistent, and quasi-transitive relation. The axiom that we employ adapts the compatibility property alluded to above to the case of Suzumura consistent and quasi-transitive greatest-element rationalizability. Because there is no such thing as a quasi-transitive closure, the condition is based on the minimal quasi-transitive extension of the Suzumura consistent closure instead. That the resulting requirement cannot be necessary follows from the observation that the minimal quasi-transitive extension does not constitute a unique minimal way of defining a quasi-transitive relation that contains a given Suzumura consistent relation: as demonstrated earlier, $qe(sc(R))$ fails to possess all properties of a closure operator, even if the relation R is Suzumura consistent. Our axiom is defined as follows.

Quasi-transitive extension compatibility. For all $S \in \Sigma$ and for all $x \in S$,

$$(x, y) \in qe(sc(R_C)) \text{ for all } y \in S \;\Rightarrow\; x \in C(S).$$

The final result of this section provides a sufficient condition for a choice function to be greatest-element rationalizable by a reflexive, Suzumura consistent, and quasi-transitive relation.

Theorem 12. *Suppose that $C\colon \Sigma \to \mathcal{X}$ is a choice function. If C satisfies quasi-transitive extension compatibility, then C is greatest-element rationalizable by a reflexive, Suzumura consistent, and quasi-transitive relation.*

Proof. Suppose that C satisfies quasi-transitive extension compatibility. By Theorem 6, $qe(sc(R_C))$ is Suzumura consistent, and Theorem 7 implies that $qe(sc(R_C))$ is quasi-transitive. Define the relation R by

$$R = qe(sc(R_C)) \cup \Delta.$$

It follows immediately that R is reflexive, Suzumura consistent, and quasi-transitive; the latter two properties are consequences of the requisite properties of $qe(sc(R_C))$ and the observation that adding the pairs in the diagonal does not affect these attributes. We complete the proof by showing that R greatest-element rationalizes C. Let $S \in \Sigma$ and $x \in S$.

Suppose first that $x \in C(S)$. By definition of the direct revealed goodness relation, it follows that $(x, y) \in R_C$ for all $y \in S$. Because $R_C \subseteq sc(R_C) \subseteq qe(sc(R_C)) \subseteq R$, this implies $(x, y) \in R$ for all $y \in S$. Thus, $x \in G(S, R)$ and hence $C(S) \subseteq G(S, R)$.

Conversely, suppose that $x \in G(S, R)$. By definition, this implies

$$(x, y) \in qe(sc(R_C)) \text{ for all } y \in S \setminus \{x\}. \tag{6}$$

If $S = \{x\}$, it follows that $(x, x) \in R_C$ and hence $(x, x) \in qe(sc(R_C))$. This implies $x \in C(S)$ by quasi-transitive extension compatibility.

Now suppose that $S \neq \{x\}$.

If there exists $y \in S \setminus \{x\}$ such that $(x, y) \in R_C$, it follows that $(x, x) \in R_C$ by definition of the direct revealed goodness relation R_C. Because $R_C \subseteq qe(sc(R_C))$, it follows that $x \in C(S)$ because (6) allows us to apply quasi-transitive extension compatibility.

If $(x, y) \notin R_C$ for all $y \in S \setminus \{x\}$, suppose that there exists $y \in S \setminus \{x\}$ such that $(x, y) \in sc(R_C)$. By definition of the Suzumura consistent closure, there exists $x^1 \in X$ such that $(x, x^1) \in R_C$ and, therefore, $(x, x) \in R_C$ by definition of R_C. Together with (6), it follows again that $x \in C(S)$.

If $(x, y) \notin R_C$ and $(x, y) \notin sc(R_C)$ for all $y \in S \setminus \{x\}$, (6) implies that there exists $y \in S \setminus \{x\}$ such that $(x, y) \in R_C$ or $(x, y) \in P(tc(sc(R_C)))$. If $(x, y) \in R_C$, $x \in C(S)$ follows as above. If $(x, y) \in tc(P(sc(R_C)))$, the definitions of sc, P, and tc imply that there exists $x^1 \in X$ such that $(x, x^1) \in R_C$ and, as in the previous case, $x \in C(S)$.

Combined with the reverse set inclusion established above, it follows that $C(S) = G(S, R)$ so that R greatest-element rationalizes C. ■

Quasi-transitive extension compatibility is not necessary for the rationalizability of a choice function by a reflexive, Suzumura consistent, and quasi-transitive relation. To see that this is the case, consider the following example. Let $X = \{x, y, z, v, w\}$,

$$\Sigma = \{\{x, y, w\}, \{x, y, z, v\}, \{y, z\}, \{y, z, v\}\},$$

$C(\{x, y, w\}) = \{y\}$, $C(\{x, y, z, v\}) = \{v\}$, $C(\{y, z\}) = \{z\}$, and $C(\{y, z, v\}) = \{z, v\}$. This choice function is greatest-element rationalized by the reflexive, Suzumura consistent, and quasi-transitive relation

$$\begin{aligned} R \;=\; & \{(x, x), (x, y), (y, x), (y, y), (y, w), (z, y), (z, z), (z, v), (z, w), \\ & (v, x), (v, y), (v, z), (v, v), (v, w), (w, w)\}. \end{aligned}$$

The direct revealed goodness relation R_C that corresponds to C is given by

$$\begin{aligned} R_C \;=\; & \{(y, x), (y, y), (y, w), (z, y), (z, z), (z, v), \\ & (v, x), (v, y), (v, z), (v, v), \}. \end{aligned}$$

This relation is Suzumura consistent and, therefore, $sc(R_C) = R_C$. Because $(z, y) \in sc(R_C)$ and $(y, x) \in sc(R_C)$, it follows that $(z, x) \in tc(P(sc(R_C)))$ and hence $(z, x) \in qe(sc(R_C))$. Because $sc(R_C) \subseteq qe(sc(R_C))$, it follows that $(z, y) \in qe(sc(R_C))$, $(z, z) \in qe(sc(R_C))$, and $(z, v) \in qe(sc(R_C))$. But $z \notin C(\{x, y, z, v\})$, a contradiction to quasi-transitive extension compatibility.

5 Concluding remarks

In our view, this paper provides two (related) main contributions. First, we define and analyze the notion of a minimal quasi-transitive extension of a relation. Although this extension does not constitute a closure operator, we show that it possesses some interesting features if the underlying relation is Suzumura consistent. The minimal quasi-transitive extension is then shown to play a major role in examining notions of rationality that involve rationalizing relations that are both Suzumura consistent and quasi-transitive.

Although our application to revealed preference theory demonstrates the usefulness of minimal quasi-transitive extensions of Suzumura consistent relations, there remain various possible applications in economic theory to be explored. In particular, minimal quasi-transitive extensions may turn out to provide a powerful analytical tool in the context of collective choice. For instance, Bordes (1976) links what he refers to as the transitive closure choice function to the majority rule as applied to two-element sets, along with the conjunction of a rationality requirement and a minimality condition. It may be worthwhile to examine the possibility of extending his observations to minimal quasi-transitive extensions. As another example, Suzumura (1999) employs extension results to Paretian Bergson-Samuelson social welfare functions, and the use of minimal quasi-transitive extensions may be an interesting addition to this area of research.

References

[1] Armstrong, W.E. (1939). The determinateness of the utility function. Economic Journal, 49, 453–467.

[2] Arrow, K.J. (1951; second ed. 1963). Social Choice and Individual Values. Wiley, New York, NY.

[3] Blair, D.H., G. Bordes, J.S. Kelly, and K. Suzumura (1976). Impossibility theorems without collective rationality. Journal of Economic Theory, 13, 361–379.

[4] Bordes, G. (1976). Consistency, rationality and collective choice. Review of Economic Studies, 43, 451–457.

[5] Bossert, W., Y. Sprumont, and K. Suzumura (2005). Consistent rationalizability. Economica, 72, 185–200.

[6] Bossert, W. and K. Suzumura (2009). Rational choice on general domains. In: K. Basu and R. Kanbur (eds.). Arguments for a Better World: Essays in Honor of Amartya Sen, Vol. I: Ethics, Welfare, and Measurement. Oxford University Press, Oxford, UK. 103–135.

[7] Bossert, W. and K. Suzumura (2010). Consistency, Choice, and Rationality. Harvard University Press, Cambridge, MA.

[8] Bossert, W. and K. Suzumura (2012). Quasi-transitive and Suzumura consistent relations. Social Choice and Welfare, 39, 323–334.

[9] Cato, S. (2012). Szpilrajn, Arrow and Suzumura: concise proofs of extension theorems and an extension. Metroeconomica, 63, 235–249.

[10] Hansson, B. (1968). Choice structures and preference relations. Synthese, 18, 443–458.

[11] Luce, R. (1956). Semiorders and a theory of utility discrimination. Econometrica, 24, 178–191.

[12] Plott, C.R. (1973). Path independence, rationality, and social choice. Econometrica, 41, 1075–1091.

[13] Raiffa, H. (1968). Decision Analysis. Addison-Wesley, Reading, MA.

[14] Richter, M.K. (1966). Revealed preference theory. Econometrica, 34, 635–645.

[15] Richter, M.K. (1971). Rational choice. In: J.S. Chipman, L. Hurwicz, M.K. Richter, and H.F. Sonnenschein (eds.). Preferences, Utility, and Demand. Harcourt Brace Jovanovich, New York, NY, 29–58.

[16] Samuelson, P.A. (1938a). A note on the pure theory of consumer's behaviour. Economica, 5, 61–71.

[17] Samuelson, P.A. (1938b). A note on the pure theory of consumer's behaviour: an addendum. Economica, 5, 353–354.

[18] Sen, A.K. (1969). Quasi-transitivity, rational choice and collective decisions. Review of Economic Studies, 36, 381–393.

[19] Sen, A.K. (1970). Collective Choice and Social Welfare. Holden-Day, San Francisco, CA.

[20] Suzumura, K. (1976). Remarks on the theory of collective choice. Economica, 43, 381–390.

[21] Suzumura, K. (1983). Rational Choice, Collective Decisions, and Social Welfare. Cambridge University Press, Cambridge, MA.

[22] Suzumura, K. (1999). Paretian welfare judgements and Bergsonian social choice. Economic Journal, 109, 204–220.

[23] Szpilrajn, E. (1930). Sur l'extension de l'ordre partiel. Fundamenta Mathematicae, 16, 386–389.