Discussion Paper Series

Institute-Wide Joint Research Project
Methodology of Social Sciences: Measuring Phenomena and Values

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Susumu Cato
University of Tokyo

Marc Fleurbaey
Paris School of Economics

July 18, 2023

E-23-002

Institute of Social Science, University of Tokyo
[https://web.iss.u-tokyo.ac.jp/methodology/en/dp/dp/]

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Susumu Cato
Institute of Social Science
University of Tokyo
7-3-1 Hongo, Bunkyo
Tokyo 113-0033
Japan
susumu.cato@gmail.com

Marc Fleurbaey
Paris School of Economics
48 Bd Jourdan
75014 PARIS
France
marc.fleurbaey@psemail.eu

Abstract. This paper develops fair quasi-orderings, which are incomplete but consistent judgments over the set of allocations. One of the most popular quasi-orderings is the Paretian dominance relation, which has no concern about fairness. To settle down a conflict of interests between individuals, the Bergson-Samuelson orderings are commonly used, as they care about fairness. Such social orderings include strong value judgments, which might not be normatively plausible. In this study, we propose two fair social quasi-orderings: the *social nested-contour quasi-ordering* and the *pairwise nested-contour quasi-ordering*. The two quasi-orderings are in between the Paretian and the Bergson-Samuelson approaches. They can be regarded as dominance criteria over multiple attributes: they employ lower contour sets of individuals (or their intersection) and apply set inclusion to compare allocations. The canonical dominance approach over multiple attributes assumes the homogeneity of individual preferences, which makes interpersonal comparison easy. We allow individual preferences to be heterogeneous. *Journal of Economic Literature* Classification Nos.: D60; D63; D71; D11; D80

Keywords Fairness; Incompleteness; Inequality; Social choice; Well-being measurement; Heterogeneous preference

Conflict of Interest. None.

Acknowledgments: We thank Dimitrios Diamantaras for his helpful comments. Financial support from KAKENHI through grants Nos. JP18H05204, JP20H01446, JP22K01387, JP22H05083, and JP22H05086 is gratefully acknowledged. This paper is prepared with support from the Institute of Social Science, University of Tokyo, and its institute-wide joint research project, "Methodology of Social Sciences: How to Measure Phenomena and Values."

1 Introduction

Imagine that all governments of the world agree with the Rawlsian idea that supporting the worst-off or poorest person should be the first priority. However, it is quite possible that these governments have different ideas about what kind of individual is the worst-off. Regarding social spending, for example, some countries, such as Denmark, have a vast array of labor market policies, while others, such as the UK, have a relatively large level of housing subsidies. A possible source for such a variety is the differences in ideas regarding the measurement of well-being.

To understand our point, consider the money metric developed by Samuelson (1974). The money metric is a measurement of well-being defined as the minimal expenditure to achieve a utility level associated with a bundle. The worst-off can be identified according to money metrics. However, the value of the money metric totally depends on a reference price, which should be chosen in advance. One could think, as in Neary (2004), of relying on averaging methods such as those developed in the "purchasing power parity" (PPP) literature. However, since there are various methods of constructing PPP prices (see Deaton and Heston [2010]), this does not provide a definitive answer regarding the choice.[1]

In general, a reference point (price, bundle, or possibly other things) is needed to make a concrete well-being measurement from information about ordinal preferences. A problem is whether a definite "just" (or "normatively plausible") reference point exists. Moreover, such reference points can depend on the characteristics of economies (preferences, technologies, cultures, historical paths) if they exist. This implies that some types of comparisons (across time or regions) are very difficult as it is likely that economies do not share their reference points. A possible way to overcome this difficulty is to allow well-being measurements to be incomplete in order to make them secure from any choice of reference points.

Incompleteness means that social evaluations are based on a social quasi-ordering (transitive and reflexive relation). In a series of works, Sen (1985, 2004, 2017) emphasizes the legitimacy of a quasi-ordering approach. According to him, incompleteness is a plausible answer for some cases and a quasi-ordering is what can be only hoped for in practice. For instance, there is an attempt of a well-being measurement, the Better Life Index, published by the OECD in 2011. While it incorporates various attributes, the weights over them are not fixed. The choice of the weights is up to people who can compare outputs associated with various weights, in which case, incompleteness arises. One way of imagining a possible use of this Better Life Index is constructing a quasi-ordering by taking the intersection across various sets of weights.

This paper develops the analysis of social quasi-orderings in a standard economic environment, where individuals have regular ordinal preferences over multiple attributes. Notably, the most influential quasi-ordering is the Paretian judgment, which requires that moving to some allocation

[1] Most of this paper focuses on a reference-bundle approach; however, our argument is also relevant to a reference-price approach. See Section 3.3.

is improving if no one is worse-off and someone is better-off. No interpersonal comparison is applied in the Paretian approach. Because of this nature, there is no way to judge who is the worst-off. In other words, the Paretian quasi-ordering completely ignores fairness.

Contrary to the Paretian judgment, a Bergson-Samuelson social welfare function yields a complete judgment over allocations. Such a function is regarded as an extension of the Paretian quasi-ordering, and it can incorporate not only efficiency but also fairness. Recently, Fleurbaey (2007) and Fleurbaey and Maniquet (2008, 2011) have develop an axiomatic approach to fair social orderings.[2] In particular, they examine two particular types of social orderings: a class of *Pazner-Schmeidler social orderings* and one of *Egalitarian Walrasian social orderings*. The two types are associated with egalitarian equivalence and envy-freeness, respectively.[3] Fair social orderings are regarded as Bergson-Samuelson's social welfare functions with concrete structures.

Notably, there is a huge "gap" between the Paretian quasi-ordering and the fair social orderings. There is no concern about fairness under the Paretian approach, while adopting a particular fair social ordering implies a strong commitment to a particular type of fairness/comparison regarding the measurement of well-being. In this paper, we develop social quasi-orderings that are in between the Paretian approach and the fair social orderings. Our proposal can be regarded as a mid-way between the two approaches, in the sense that it does introduce fairness considerations but without committing to specific interpersonal comparisons. Two types of social quasi-orderings are examined in detail: the *social nested-contour quasi-ordering* and the *pairwise nested-contour quasi-ordering*. The intersection of lower-contour sets of individuals is crucial information for the social nested-contour quasi-ordering. That is, if the intersection of lower-contour sets of individuals under state B contains that under state A, then a move from A to B is a social improvement. The social nested-contour quasi-ordering has a close link to the Pazner-Schmeidler social orderings and Egalitarian Walrasian social orderings. This quasi-ordering sheds light on the connection between egalitarian equivalence and envy-freeness.

The pairwise nested-contour quasi-ordering is weaker (more partial) than the social nested-contour quasi-ordering. If an individual's upper-contour set has no intersection with another individual's lower-contour set, then the former's well-being dominates the latter's. According to the pairwise nested-contour quasi-ordering, if, in state A, each individual is dominating some individual in state B, a move from B to A is a social improvement.

The two criteria defined above are in a particular relationship with the existing criteria. Consider a move from some state to another. The following summarizes the relationship among these

[2] Applications of the approach can be found in Fleurbaey and Gaulier (2009), Decancq et al. (2015a), and Jones and Klenow (2016).

[3] Envy-freeness is a concept developed by, Tinbergen (1946), Foley (1967), Kolm (1971), and Varian (1974, 1976). An allocation is said to be envy-free if no individual strictly prefers any other individual's bundle to his own bundle. The concept of egalitarian equivalence was introduced by Pazner and Schmeidler (1978). An allocation is said to be egalitarian equivalent if there exists a hypothetical bundle such that each individual is indifferent toward choosing between that and his own bundle. These classical works mainly focus on identifying the set of equitable and efficient allocations: two inequitable allocations cannot be compared in their approach.

criteria:

- If the Paretian quasi-ordering approves this move, so does the pairwise nested-contour quasi-ordering;
- If the pairwise nested-contour quasi-ordering approves this move, so does the social nested-contour quasi-ordering;
- If the social nested-contour quasi-ordering approves this move, so do the Pazner-Schmeidler social orderings and the Egalitarian Walrasian social orderings.

We show that differences between the aforementioned quasi-orderings and orderings are based on different ideas of the set of worst-off individuals.

In this paper, we provide axiomatic characterizations of the two quasi-orderings. The pairwise nested-contour quasi-ordering is characterized by three weak forms of equity, informational parsimony, and efficiency (the Pareto principle, Hansson independence, and equity among equals). Each of them is a basic axiom in the literature. By introducing a new axiom, we also characterize the social nested-contour quasi-ordering.

Our approach may be useful for implementing international comparisons of living standards. Nations have different fundamental parameters, such as endowments, technologies, or geographic structures. Such parameters are crucial for constructing complete well-being measurements. One approach is taking an endogenously fixed parameter as a reference point — as suggested by Fleurbaey and Maniquet (2011) and Fleurbaey and Tadenuma (2014). Another approach is choosing some country as a reference point. For example, Jones and Klenow (2016) make a quantitative cross-country comparison by using the status of the U.S. as a reference. Such choices of reference points strongly matter: results can change by modifying a reference point. By contrast, the social and pairwise nested-contour quasi-orderings are independent of reference choices. This is a benefit of accepting incompleteness, as the judgments based on such quasi-orderings may appear normatively more secure.

The core of our fair quasi-orderings is a general construction of well-being measurements. Fleurbaey and Maniquet (2017) provide an axiomatic analysis for well-being measurements using the information on individual preferences. Indifference curves and lower contour sets are employed in both their work and this study. In Fleurbaey and Maniquet's framework, well-being measurements are supposed to be complete, and there is no aggregation procedure for individual well-beings. Our study considers an incomplete evaluation of individual well-being and has an aggregation procedure with equity axioms. Bosmans, Decancq, and Ooghe (2018) examine a social welfare evaluation based on money metrics. They introduce a new equity axiom, which utilizes information about Scitovsky curves: a transfer among equals leads to a social improvement when the Scitovsky curve does not change. Their new axiom is substantially weaker than the equity axiom that we use in this paper. Piacquadio (2017) proposes a general class of social welfare orderings, including money metric approaches. He employs equity among equals to derive his general class and derives a utilitarian form. One key difference is that our study imposes Hansson independence on incomplete social evaluations, while Piacquadio (2017) does not impose any independence axiom, considering a complete social ordering.

The rest of this paper is organized as follows. Section 2 introduces our setting. Section 3 presents our two quasi-orderings and a fundamental analysis of their properties. Section 4 conducts the axiomatic analysis and main characterizations. Section 5 discusses another approach to fair quasi-orderings.

2 Setting

Let N be the set of individuals. Assume that N is finite. There exist ℓ (≥ 2) commodities: all of them are private goods. The consumption set is $X = \mathbb{R}^{\ell}_{+}$. Individual i's consumption bundle is an element of X. An allocation is a list of individual bundles: a typical allocation is denoted by $x_N = (x_i)_{i \in N} \in X^N$.

Each individual $i \in N$ has a preference ordering R_i over X. The asymmetric and symmetric parts of R_i are denoted by P_i and I_i, respectively. Preferences are assumed to be continuous, monotonic,[4] and convex. Let $\mathcal{R}$ be the set of preferences over X satisfying these three properties. An economy e is defined by the preference profile $R_N = (R_i)_{i \in N} \in \mathcal{R}^N$. The domain is defined as follows:

$$\mathcal{E} = \bigcup_{N \subseteq \mathbb{N}: N < \infty \ \& \ N \neq \emptyset} \mathcal{R}^N.$$

A *social ordering function (SOF)* R is a mapping from $\mathcal{E}$ to the set of orderings over the set X^N of allocations. A *social quasi-ordering function (SQF)* is a mapping from $\mathcal{E}$ to the set of quasi-orderings over the set X^N of allocations. The asymmetric and symmetric parts of $R(e)$ are denoted by $P(e)$ and $I(e)$, respectively. An SQF R is a *refinement* of R' if $R'(e) \subseteq R(e)$ for all $e \in \mathcal{E}$. An SQF R is an *extension* of R' if it is a refinement such that, additionally, $P'(e) \subseteq P(e)$ for all $e \in \mathcal{E}$. Each of a refinement and an extension yields a quasi-ordering that is compatible with the original one.

The lower-contour set and upper-contour set are introduced as follows:

$$L(x_i, R_i) = \{y \in X : x_i R_i y\} \text{ and } U(x_i, R_i) = \{y \in X : y R_i x_i\}.$$

Now, we introduce major fair social orderings. Pick some reference bundle $\Omega \in \mathbb{R}^{\ell}_{++}$. Define $u^{\Omega}_{R} : X \times \mathcal{R} \to \mathbb{R}$ as follows:

$$u^{\Omega}_{R}(x_i, R_i) = \lambda \Leftrightarrow x_i I_i \lambda \Omega.$$

[4] The ordering R_i is monotonic if $x R_i y$ whenever $x \geq y$ and $x P_i y$ whenever $x \gg y$, and strictly monotonic if it is monotonic and in addition, $x P_i y$ whenever $x > y$.

Note that $u_R^\Omega(\cdot, R_i)$ is a utility representation of R_i. The *Pazner-Schmeidler SOF* is defined as follows:

$$x_N R_{PS}^\Omega y_N \Leftrightarrow \min_{i\in N} u_R^\Omega(x_i, R_i) \geq \min_{i\in N} u_R^\Omega(y_i, R_i).$$

The Pazner-Schmeidler SOF is associated with the concept of egalitarian equivalence, defined in the introduction. Formally, for any egalitarian equivalent allocation $x_N \in X^N$, there exists $z \in X$ such that $x_i I_i z$ for all $i \in N$. Given any Ω, the optimal allocation with respect to R_{PS}^Ω must be egalitarian equivalent and efficient.

Define $u_p^\Omega : X \times \mathcal{R} \to \mathbb{R}$ as follows:

$$u_p^\Omega(x_i, R_i) = \frac{1}{p\Omega} \min\{pz : zR_i x_i \quad z \in X\}.$$

The *Egalitarian Walras SOF* is defined as follows:

$$x_N R_{EW}^\Omega y_N \Leftrightarrow \max_p \min_{i\in N} u_p^\Omega(x_i, R_i) \geq \max_p \min_{i\in N} u_p^\Omega(y_i, R_i).$$

The Egalitarian Walras SOF is associated with the concept of envy-freeness: an allocation is said to be envy-free if everyone weakly prefers his own bundle to others' bundles. If Ω is equal to an aggregate endowment of the economy, the optimal allocation with respect to R_{EW}^Ω must be a Walrasian allocation with equal income, and must therefore be envy-free and efficient.

3 Two Fair Social Quasi-Orderings

3.1 Definitions and relations

In this section, we introduce fair social quasi-orderings and provide a basic analysis of their characteristics. Given $R_N \in \mathcal{R}^N$, let

$$\mathcal{L}(x_N, R_N) = \bigcap_{i\in N} L(x_i, R_i).$$

That is, x_i is at least good as $x \in \mathcal{L}(x_N, R_N)$ for every individual $i \in N$. Given $R_N \in \mathcal{R}^N$, let

$$\mathcal{U}(x_N, R_N) = \bigcup_{i\in N} U(x_i, R_i).$$

For each $x \in \mathcal{U}(x_N, R_N)$, there is some individual $i \in N$ who weakly prefers x to x_i. Both $\mathcal{L}(x_N, R_N)$ and $\mathcal{U}(x_N, R_N)$ are closed sets. We call $\mathcal{L}(x_N, R_N)$ the *social lower-contour set at* x_N (resp. $\mathcal{U}(x_N, R_N)$ the *social upper-contour set at* x_N). Note that the social lower-contour set (resp. the social upper-contour set) cannot be the lower-contour (upper-contour) set for social preference. The lower-contour (upper-contour) set for social preference must be a subset of the set of allocations, X^N, while the social lower-contour set or the social upper-contour set is a subset of the consumption set, X.

Note that $L(x_i, R_i) = X - \text{Int}(U(x_i, R_i))$. Therefore,

$$\begin{aligned}\mathcal{L}(x_N, R_N) &= \bigcap_{i \in N} \Big(X - \text{Int}(U(x_i, R_i))\Big), \\ &= X - \text{Int}\Big(\mathcal{U}(x_N, R_N)\Big).\end{aligned}$$

The *Social Nested-Contour SQF* is defined as follows:

$$x_N R_S(e) y_N \Leftrightarrow \mathcal{L}(x_N, R_N) \supseteq \mathcal{L}(x_N, R_N).$$

According to this SQF, an allocation is socially at least as good as another allocation if the social lower-contour set at x_N includes the social lower-contour set at y_N. That is, the set inclusion relation works as a dominance relation. It is easy to see that $R_S(e)$ is always a quasi-ordering.

The social nested-contour SQF can equivalently be defined in terms of the social upper-contour set:

$$x_N R_S(e) y_N \Leftrightarrow \mathcal{U}(x_N, R_N) \subseteq \mathcal{U}(y_N, R_N),$$

because

$$\begin{aligned}\mathcal{L}(x_N, R_N) \supseteq \mathcal{L}(y_N, R_N) &\Leftrightarrow X - \text{Int}\Big(\mathcal{U}(x_N, R_N)\Big) \supseteq X - \text{Int}\Big(\mathcal{U}(y_N, R_N)\Big) \\ &\Leftrightarrow \mathcal{U}(x_N, R_N) \subseteq \mathcal{U}(y_N, R_N).\end{aligned}$$

As a notable feature, the social nested-contour SQF is associated with the concept of *envy-free equivalence* by Pazner (1977), which requires that there exists a hypothetical envy-free allocation where the consumption bundle in the original allocation is indifferent to the hypothetical one for each individual.[5] Note that any egalitarian-equivalent allocation must be envy-free equivalent, and so is any envy-free allocation. Now, we consider a simple exchange economy with an aggregate endowment, in which all commodities are transferable. Given this endowment, we can get an optimal allocation with respect to $R_S(e)$. That is, an allocation x_N is said to be optimal with respect to $R_S(e)$ if there exists no feasible allocation y_N such that $y_N P_S(e) x_N$. An example of an optimal allocation is shown in Figure 1 (the bundles x_i are not shown on the figure, only the indifference curves are shown). Under any optimal allocation for $R_S(e)$, each individual's upper

[5] *Undominated diversity* introduced by Van Parijs (1990) also uses information about the intersection of individuals' lower contour set. According to him, an individual i is worse-off than another individual j if individual j's consumption is better than individual i's consumption for everyone: i.e., x_i is in the interior of $\bigcap_{k \in N} L(x_j, R_k)$. Undominated diversity requires that there is no such dominance relation. However, undominated diversity and envy-free equivalence (the social nested-contour SQF) are independent.

contour set has an intersection with the social lower-contour set. This means that we can find a hypothetical envy-free (but not necessarily efficient) allocation over the social lower-contour set. Here, (z_1, z_2, z_3) is an example of a hypothetical envy-free allocation in Figure 1.

The *Pairwise Nested-Contour SQF* is defined as follows:

$$x_N R_P(e) y_N \Leftrightarrow \forall i \in N, \exists j \in N, L(x_i, R_i) \supseteq L(y_j, R_j).$$

R_P always generates a quasi-ordering over the set of allocations. It is easy to see that $R_P(e)$ is reflexive for any $R_N \in \mathcal{R}^N$. Then, let us check transitivity. Suppose that $x_N R_P(e) y_N$ and $y_N R_P(e) z_N$. Take any $i \in N$. By definition, there exists $j \in N, L(x_i, R_i) \supseteq L(y_j, R_j)$. Since $y_N R_P(e) z_N$, there exists $k \in N, L(y_j, R_j) \supseteq L(z_k, R_k)$. Note that $L(x_i, R_i) \supseteq L(y_j, R_j)$ and $L(y_j, R_j) \supseteq L(z_k, R_k)$. Thus, $L(x_i, R_i) \supseteq L(z_k, R_k)$. This means that

$$\forall i \in N, \exists k \in N, L(x_i, R_i) \supseteq L(z_k, R_k).$$

Therefore, we have $x_N R_P(e) z_N$.

Let us consider an optimal allocation in a simple exchange economy with respect to the pairwise nested-contour SQF. Under any optimal allocation, each individual's upper contour set has an intersection with the lower-contour set of every individual. An example of an optimal allocation in some economy is shown in Figure 2 (again, without the bundles x_i). This means that for all distinct $j, k \in N$, there exists $z(j,k) \in X$ such that $z(j,k) I_i x_i$ for all $i \in \{j, k\}$. An allocation satisfying this property is called *pairwise egalitarian equivalent*. In Figure 2, z_1 is the bundle associated with individuals 1 and 3; z_2 is the bundle associated with individuals 2 and 3; z_3 is the bundle associated with individuals 1 and 2. Pairwise egalitarian equivalence is weaker than envy-free equivalence, which is weaker than envy-freeness or egalitarian equivalence.

The following result shows a hierarchical refinement structure among the three criteria.

Proposition 1.
(i) R_S is a refinement of R_P;
(ii) R^{Ω}_{PS} is a refinement of R_S;
(iii) R^{Ω}_{EW} is a refinement of R_S.

Proof. (i) Let $e = R_N \in \mathcal{E}$. Suppose that $x_N R_P(e) y_N$. By definition, we have $\forall i \in N, \exists j \in N, L(x_i, R_i) \supseteq L(y_j, R_j)$. By way of contradiction, suppose that $x^* \in \mathcal{L}(y_N, R_N)$ but $x^* \notin \mathcal{L}(x_N, R_N)$. Then, $x^* \notin L(x_{i^*}, R_{i^*})$ for some $i^* \in N$. By our assumption, there exists $j \in N$ such that $L(x_{i^*}, R_{i^*}) \supseteq L(y_j, R_j)$. Then, $x^* \notin L(y_j, R_j)$, which implies that $x^* \notin \mathcal{L}(y_N, R_N)$. This is a contradiction.

(ii) Suppose that $x_N R_S(e) y_N$ but $\neg(x_N R^{\Omega}_{PS}(e) y_N)$. Then, $\mathcal{L}(x_N, R_N) \supseteq \mathcal{L}(y_N, R_N)$ and

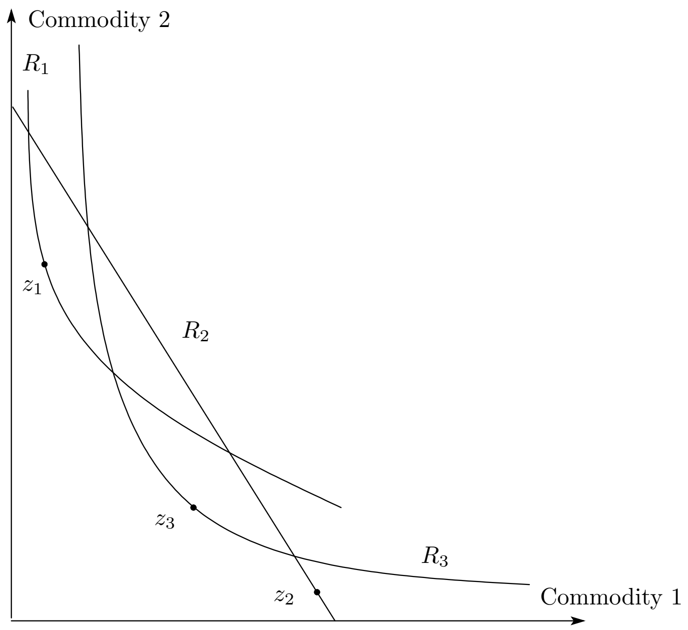

Figure 1: The social nested-contour SQF

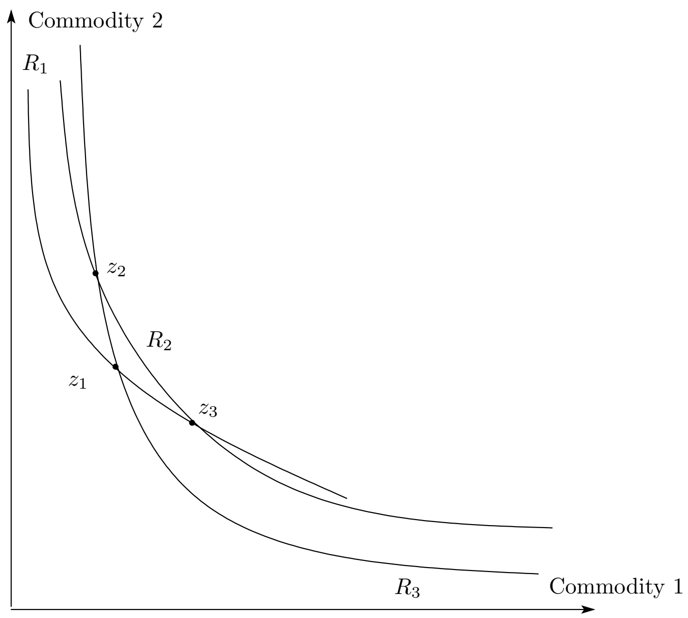

Figure 2: The pairwise nested-contour SQF

$\min_{i\in N} u_R^\Omega(y_i, R_i) > \min_{i\in N} u_R^\Omega(x_i, R_i)$. Let

$$i_0 \in \arg\min_{i\in N} u_\Omega(x_i, R_i).$$

We have $u_R^\Omega(x_{i_0}, R_{i_0}) < u_R^\Omega(y_i, R_i)$ for all $i \in N$. This implies that $u_R^\Omega(x_{i_0}, R_{i_0})\Omega \in \mathrm{Int}(L(y_i, R_i))$ for all $i \in N$. Thus, $u_R^\Omega(x_{i_0}, R_{i_0})\Omega \in \mathrm{Int}(\mathcal{L}(y_N, R_N))$. Since $u_R^\Omega(x_{i_0}, R_{i_0})\Omega\, I_{i_0}\, x_{i_0}$, one has $u_R^\Omega(x_{i_0}, R_{i_0})\Omega \in \mathcal{U}(x_N, R_N)$. But it is impossible to simultaneously have

$$\mathrm{Int}(\mathcal{L}(y_N, R_N)) \cap \mathcal{U}(x_N, R_N) \neq \emptyset$$

and $\mathcal{L}(x_N, R_N) \supseteq \mathcal{L}(y_N, R_N)$.

(iii) Suppose that $x_N R_S(e) y_N$ but $\neg(x_N R_{EW}^\Omega(e) y_N)$. Since $\mathcal{L}(x_N, R_N) \supseteq \mathcal{L}(y_N, R_N)$, we have $\mathcal{U}(x_N, R_N) \subseteq \mathcal{U}(y_N, R_N)$. By monotonicity of the convex hull, $co\big(\mathcal{U}(x_N, R_N)\big) \subseteq co\big(\mathcal{U}(y_N, R_N)\big)$, where co denotes the convex hull. However, $\max_p \min_{i\in N} u_p^\Omega(x_i, R_i) < \max_p \min_{i\in N} u_p^\Omega(y_i, R_i)$. Note that $\max_p \min_{i\in N} u_p^\Omega(x_i, R_i)$ is $\min\{r : r\Omega \in co\big(\mathcal{U}(x_N, R_N)\big)\}$. Thus, $\min\{r : r\Omega \in co\big(\mathcal{U}(x_N, R_N)\big)\} < \min\{r : r\Omega \in co\big(\mathcal{U}(y_N, R_N)\big)\}$. This violates the fact that $co\big(\mathcal{U}(x_N, R_N)\big) \subseteq co\big(\mathcal{U}(y_N, R_N)\big)$. ■

R_S is not an extension of R_P. There exists $e = R_N \in \mathcal{E}$ and $x_N, y_N \in N$ such that

$$x_N P_P(e) y_N \text{ and } x_N I_S(e) y_N.$$

See Figure 3. It is easy to check that $(x_1, x_2, x_3')P_P(e)(x_1, x_2, x_3)$ and $(x_1, x_2, x_3')I_S(e)(x_1, x_2, x_3)$.

Moreover, R_{PS}^Ω is not an extension of R_S because one can have $x_N P_S(e) y_N$ without having $x_N P_{PS}^\Omega(e) y_N$ because the part where $\mathcal{L}(x_N, R_N)$ dominates $\mathcal{L}(y_N, R_N)$ may not be in the direction of Ω.

The following result shows that there exists no difference between the two social quasi-ordering for the two-person case. This result is quite intuitive because there exists no difference between egalitarian equivalence and pairwise egalitarian equivalence in the two-person case.

Proposition 2. If there are only two individuals in an economy e, then $R_S(e)$ is identical with $R_P(e)$.

Proof. By Proposition 1, $R_P(e) \subseteq R_S(e)$. We need to show the converse. Suppose that $x_N R_S(e) y_N$. Then, $L(x_1, R_1) \cap L(x_2, R_2) \supseteq L(y_1, R_1) \cap L(y_2, R_2)$. Note that $x_1 R_1 y_1$ or $x_2 R_2 y_2$. Without loss of generality, we assume that $x_1 R_1 y_1$. If $x_2 R_2 y_2$, then $x_N R_P(e) y_N$. Assume that $y_2 P_2 x_2$. Since $x_1 R_1 y_1$, it follows that $L(x_1, R_1) \supseteq L(y_1, R_1)$. Since $U(y_2, R_2) \cap L(x_2, R_2) = \emptyset$, we need to show that $L(x_2, R_2) \supseteq L(y_1, R_1)$. By way of contradiction, assume that $L(x_2, R_2) \not\supseteq L(y_1, R_1)$. Since $U(y_2, R_2) \cap L(x_2, R_2) = \emptyset$, $L(x_2, R_2) \not\supseteq L(y_1, R_1) \cap L(y_2, R_2)$. This contradicts the assumption that $L(x_1, R_1) \cap L(x_2, R_2) \supseteq L(y_1, R_1) \cap L(y_2, R_2)$, which implies that $L(x_1, R_1) \supseteq L(y_1, R_1) \cap L(y_2, R_2)$. ■

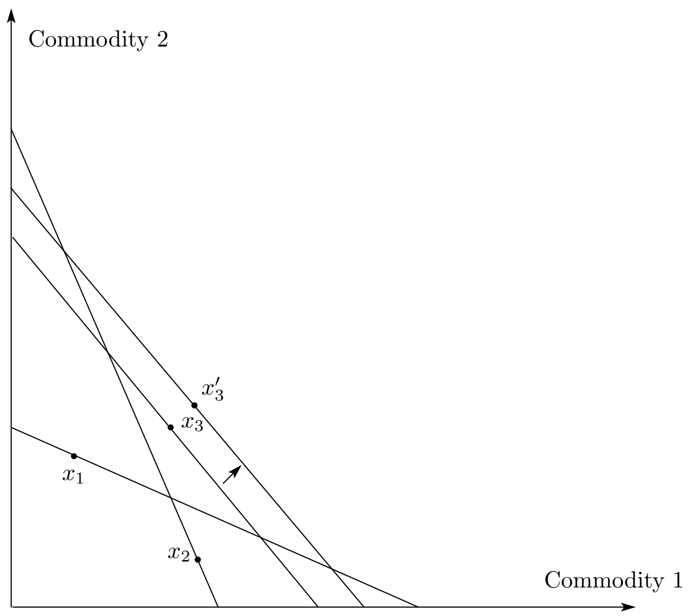

Figure 3: R_S is not an extension of R_P

A fundamental approach to a quasi-ordering consists in taking the intersection of orderings (the *intersection approach*). It is known that the intersection of orderings is always a quasi-ordering. As shown by Donaldson and Weymark (1998), in general, a quasi-ordering is the intersection of all ordering extensions (i.e., orderings which are compatible with the original quasi-ordering). Since $R_S(e)$ is a quasi-ordering, it must be the intersection of *all* ordering extensions. By this process, we can implicitly characterize the social nested-contour SQF. A fundamental difficulty of this procedure is that the construction is not explicit in the sense that ordering extensions for each $R_S(e)$ are obtained through Zorn's lemma (or the axiom of choice).

Fortunately, we can obtain an explicit characterization of R_S by taking the intersection of a *particular* set of orderings, which can be explicitly formulated. Given a reference subset $\Theta \subseteq X$ of the consumption set, we can construct an SQF by taking the intersection:[6]

$$R_\Theta = \bigcap_{\Omega \in \Theta} R^{\Omega}_{PS}.$$

It is easy to see that $R_\Theta \subseteq R_{\Theta'}$ if $\Theta \supseteq \Theta'$. The following result states that the social nested-contour SQF is characterized as the intersection of all Pazner-Schmeidler SOFs.

[6]This construction process is related with work on partial comparability of Blackorby (1975). He focuses on the case of utilitarian sum and he proposes a procedure to construct a reference set Θ.

Theorem 1. On the subdomain of $\mathcal{R}$ in which preferences are strictly monotonic,

$$R_S(e) = \bigcap_{\Omega \in \mathbb{R}^{\ell}_{++}} R^{\Omega}_{PS}(e).$$

Proof. We have $x_N R_S(e) y_N \Rightarrow x_N R^{\Omega}_{PS}(e) y_N$ for all $e \in \mathcal{R}$ and all $\Omega \in \mathbb{R}^{\ell}_{++}$, and thus, it is obvious that

$$R_S(e) \subseteq \bigcap_{\Omega \in \mathbb{R}^{\ell}_{++}} R^{\Omega}_{PS}(e).$$

We must show that

$$R_S(e) \supseteq \bigcap_{\Omega \in \mathbb{R}^{\ell}_{++}} R^{\Omega}_{PS}(e)$$

when preferences are strictly monotonic. By way of contradiction, assume that there exist $x_N, y_N \in X^N$ and $e = R_N \in \mathcal{R}^N$ such that

$$(x_N, y_N) \in \bigcap_{\Omega \in \mathbb{R}^{\ell}_{++}} R^{\Omega}_{PS}(e) \text{ and } \neg(x_N R_S(e) y_N).$$

Then, we must have the following: $x_N R^{\Omega}_{PS}(R_N) y_N$ for all $\Omega \in \mathbb{R}^{\ell}_{++}$ and $\mathcal{L}(x_N, R_N) \not\supseteq \mathcal{L}(y_N, R_N)$. Since preferences are strictly monotonic, there exists $\Omega_0 \in \mathbb{R}^{\ell}_{++}$ such that $\Omega_0 \in \mathcal{L}(y_N, R_N)$ and $\Omega_0 \notin \mathcal{L}(x_N, R_N)$. Note that $y_N P^{\Omega_0}_{PS}(R_N) x_N$. This is a contradiction. ■

When preferences are not strictly monotonic, it is possible to have $x_N P_S(e) y_N$ while $x_N I^{\Omega}_{PS}(e) y_N$ for all $\Omega \in \mathbb{R}^{\ell}_{++}$ and even for all $\Omega \in X$.

Let us now define the *Social Nested-Contour Rule** as follows:

$$x_N \widehat{R}_S(e) y_N \Leftrightarrow \mathcal{U}(x_N, R_N) \cap \mathcal{L}(y_N, R_N) = \emptyset.$$

and the *Pairwise Nested-Contour Rule** as follows:

$$x_N \widehat{R}_P(e) y_N \Leftrightarrow \forall i \in N, \exists j \in N, U(x_j, R_j) \cap L(y_i, R_i) = \emptyset.$$

Since $U(x_i, R_i) = X - \mathrm{Int}(L(x_i, R_i))$, when $U(x_j, R_j) \cap L(y_i, R_i) = \emptyset$, necessarily

$$\mathrm{Int}(L(x_j, R_j)) \supseteq L(y_i, R_i).$$

Therefore, an equivalent expression is as follows:

$$x_N \widehat{R}_P(e) y_N \Leftrightarrow \forall i \in N, \exists j \in N, \mathrm{Int}(L(x_i, R_i)) \supseteq L(y_j, R_j).$$

Similarly, we can obtain the following equivalent expression of $\widehat{R}_S(e)$:

$$x_N \widehat{R}_S(e) y_N \Leftrightarrow \mathrm{Int}\Big(\mathcal{L}(x_N, R_N)\Big) \supseteq \mathcal{L}(y_N, R_N).$$

Both the Social Nested-Contour Rule* and the Pairwise Nested-Contour Rule* generate transitive and asymmetric social preferences. The Social Nested-Contour SQF and the Pairwise Nested-Contour SQF are extensions of $\widehat{R}_S$ and $\widehat{R}_P$, respectively.

The following result shows that the variants introduced above are compatible with each other and also that they are compatible with R^Ω_{PS} and R^Ω_{EW}.

Proposition 3.
(i) $\widehat{R}_S$ is an extension of $\widehat{R}_P$;
(ii) R^Ω_{PS} is an extension of $\widehat{R}_S$;
(iii) R^Ω_{EW} is an extension of $\widehat{R}_S$.

Proof. Let $e = R_N \in \mathcal{E}$. Since $\widehat{R}_S(e)$ and $\widehat{R}_P(e)$ is asymmetric, it suffices to show that $x_N \widehat{R}_P(e) y_N \Rightarrow x_N R^\Omega_S y_N$. Suppose that $x_N R_P(e) y_N$. Then,

$$\forall i \in N, \exists j \in N, U(x_i, R_i) \cap L(y_j, R_j) = \emptyset. \tag{1}$$

Suppose that $\mathcal{L}(y_N, R_N) \cap \mathcal{U}(x_N, R_N) \neq \emptyset$. Note that

$$\bigcap_{i \in N} L(y_i, R_i) \cap \bigcup_{i \in N} U(x_i, R_i) \neq \emptyset \Rightarrow \bigcap_{i \in N} L(y_i, R_i) \cap U(x_{i^*}, R_{i^*}) \neq \emptyset \text{ for some } i^* \in N$$

Therefore,

$$L(y_i, R_i) \cap U(x_{i^*}, R_{i^*}) \neq \emptyset \text{ for all } i \in N.$$

This contradicts (1).

(ii) Since $\widehat{R}_S(e)$ is asymmetric, it suffices to show that $x_N \widehat{R}_S(e) y_N \Rightarrow x_N P^\Omega_{PS} y_N$. Suppose that $x_N \widehat{R}_S(e) y_N$ but $\neg(x_N P^\Omega_{PS} y_N)$. Then, $\min_{i \in N} u^\Omega_R(y_i, R_i) \geq \min_{i \in N} u^\Omega_R(x_i, R_i)$. Let

$$i_0 \in \arg\min_{i \in N} u^\Omega_R(x_i, R_i).$$

We have $u^\Omega_R(x_{i_0}, R_{i_0}) \leq u^\Omega_R(y_i, R_i)$ for all $i \in N$. This implies that $u^\Omega_R(x_{i_0}, R_{i_0})\Omega \in L(y_i, R_i)$ for all $i \in N$. Note that $u^\Omega_R(x_{i_0}, R_{i_0})\Omega \in U(x_{i_0}, R_{i_0})$. Thus,

$$u^\Omega_R(x_{i_0}, R_{i_0})\Omega \in \mathcal{L}(y_N, R_N) \text{ and } u^\Omega_R(x_{i_0}, R_{i_0})\Omega \in \mathcal{U}(x_N, R_N).$$

This is a contradiction.

(iii) This can be proved in a similar way to (ii). ■

Figure 4 shows the relationship among social rules. This figure includes the Paretian SQF and the Paretian Rule*, which are defined, respectively, as follows:[7]

$$
\begin{aligned}
x_N R_R(e) y_N &\Leftrightarrow \forall i \in N : L(x_i, R_i) \supseteq L(y_i, R_i),\\
x_N \widehat{R}_R(e) y_N &\Leftrightarrow \forall i \in N : U(x_i, R_i) \cap L(y_i, R_i) = \emptyset.
\end{aligned}
$$

We can see that our rules are intermediate between the traditional Paretian approach and the fair-ordering approach.

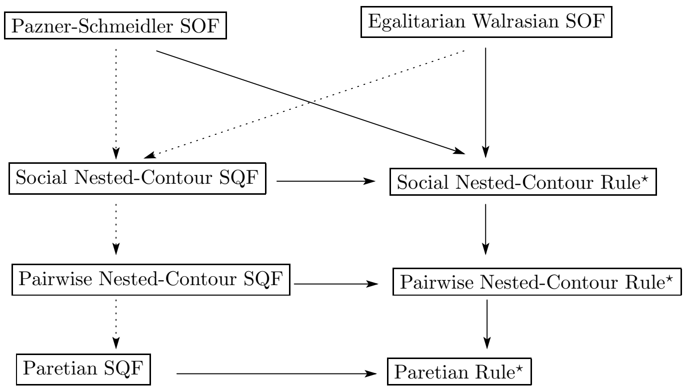

Figure 4: Relationship: "A→B" means "A is an extension of B" and a dashed line means "A is a refinement of B"

To further examine the difference between $R_S(e)$ and $\widehat{R}_S(e)$, we provide remarks on continuity. Now, a rule R is said to be *open* if for all $e = R_N \in \mathcal{E}$ and $x_N \in X^N$, both $\{z_N \in X^N : z_N P(e) x_N\}$ and $\{z_N \in X^N : x_N P(e) z_N\}$ are open; *closed* if for all $e = R_N \in \mathcal{E}$ and $x_N \in X^N$, both $\{z_N \in X^N : z_N R(e) x_N\}$ and $\{z_N \in X^N : x_N R(e) z_N\}$ are closed. A continuous rule is open and closed. In general, if preference is an ordering, there is no difference between openness and closedness. Therefore, an open SOF is continuous, and so is a closed SOF. However, openness and closedness are independent when preference is incomplete.

First, R_S is obviously closed. However, it is not open. Remember that if it is open, if y_N is socially better than x_N, for any sequence $\{y^k_N\}$ converging to y_N, we can find $\hat{k} \in \mathbb{N}$ such that $y^k_N P(e) x_N$ for all $k \geq \hat{k}$. Now, fix $x^* \in \mathbb{R}^k_{++}$. Let $x^*_N = (x^*, x^*, \ldots, x^*)$. Pick up $y^* > x^*$ and let $y^*_N = (y^*, x^*, \ldots, x^*)$. Let $e = R^N$ be such that $L(x^*, R_1)$ is not identical with $L(x^*, R_i)$ for some $i \neq 1$. It is clear that $y^*_N P(e) x^*_N$. Take a sequence $\{y^k_N\}$ such that

$$y^k_1 = y^* \Big(1 + \frac{1}{k}\Big) \text{ and } y^k_i = x^* \Big(1 - \frac{1}{k}\Big).$$

There exists no $\hat{k} \in \mathbb{N}$ such that $y^k_N P(e) x^*_N$ for all $k \geq \hat{k}$.

[7] $R_R(e)$ corresponds to strong Pareto, while $\widehat{R}_R(e)$ corresponds to weak Pareto. These axioms will be introduced in the next section.

Second, $\widehat{R}_S$ is open but not closed. As a consequence, neither of them is continuous. This point is closely related with the work of Schmeidler (1971), who shows that any non-trivial and continuous quasi-ordering must be complete. This implies that there exists no "incomplete" fair quasi-ordering that is continuous (open and closed).

3.2 Who is worst-off?

Each equitable criterion offers a specification of the set of worst-off agents. Under the Pazner-Schmeidler SOF, the worst-off agent is an individual who attains the minimum value with respect to $u_R^\Omega(x_i, R_i)$. To be precise, the set of worst-off agents with respect to the Pazner-Schmeidler SOF is $\{i \in N : u_R^\Omega(x_i, R_i) = \min_j u_R^\Omega(x_j, R_j)\}$. When we compare two allocations, any changes in bundles of individuals who are not worst-off do not matter.

Under the social nested-contour SQF, worst-off agents are individuals whose upper-contour set has an intersection with the social lower-contour set. More formally, the set of worst-off agents can be specified as follows:

$$\mathcal{S}_S(x_N, R_N) = \{i \in N : \mathcal{L}(x_N, R_N) \cap U(x_i, R_i) \neq \emptyset\}.$$

It is easy to see that

$$\mathcal{L}(y_N, R_N) \supseteq \mathcal{L}(x_N, R_N) \text{ if and only if } \bigcap_{i \in \mathcal{S}_S(x_N, R_N)} L(y_i, R_i) \supseteq \bigcap_{i \in \mathcal{S}_S(y_N, R_N)} L(x_i, R_i).$$

An individual who does not have an intersection with the social lower-contour set does not matter for the social lower-contour set. In Figure 3, individuals 1 and 2 are worst-off, while individual 3 is better-off in both allocations, (x_1, x_2, x_3) and (x_1, x_2, x_3'). The change (from x_3 to x_3') is irrelevant to the social lower-contour set, and thus, the two allocations are socially indifferent.

Although the set of worst-off agents is not explicitly stated under the pairwise nested-contour SQF, it is defined as follows:

$$\mathcal{S}_P(x_N, R_N) = \{i \in N : U(x_i, R_i) \cap L(x_j, R_j) \neq \emptyset \text{ for all } j \in N\}.$$

Note that for each $j \notin \mathcal{S}_P(x_N, R_N)$, there exists $i \in \mathcal{S}_P(x_N, R_N)$ such that $L(x_i, R_i) \cap U(x_j, R_j) = \emptyset$.

As we can see in the following proposition, only $\mathcal{S}_P(x_N, R_N)$ matters for social judgments.

Proposition 4. The following three statements are equivalent:
(i) $\forall i \in \mathcal{S}_P(x_N, R_N), \exists j \in \mathcal{S}_P(y_N, R_N), L(x_i, R_i) \supseteq L(y_j, R_j)$ (resp. $U(x_i, R_i) \cap L(x_j, R_j) = \emptyset$);
(ii) $\forall i \in \mathcal{S}_P(x_N, R_N), \exists j \in N, L(x_i, R_i) \supseteq L(y_j, R_j)$ (resp. $U(x_i, R_i) \cap L(x_j, R_j) = \emptyset$);
(iii) $\forall i \in N, \exists j \in N, L(x_i, R_i) \supseteq L(y_j, R_j)$ (resp. $U(x_i, R_i) \cap L(x_j, R_j) = \emptyset$).

Proof. (i) $\Leftrightarrow$ (ii). Since the "$\Rightarrow$" part is obvious, it suffices to show the converse part. Suppose that (ii) is true. Take any $i \in \mathcal{S}_P(x_N, R_N)$. Then, there exists $j \in N$ such that $L(x_i, R_i) \supseteq L(y_j, R_j)$. If $j \in \mathcal{S}_P(y_N, R_N)$, the proof is complete. If $j \notin \mathcal{S}_P(y_N, R_N)$, we can find $k \in \mathcal{S}_P(y_N, R_N)$ such that $U(y_j, R_j) \cap L(y_k, R_k) = \emptyset$. Note that $L(y_j, R_j) \supseteq L(y_k, R_k)$. Since $L(x_i, R_i) \supseteq L(y_j, R_j)$ and $L(y_j, R_j) \supseteq L(y_k, R_k)$, we have $L(x_i, R_i) \supseteq L(y_k, R_k)$. This complete the proof.

(ii) $\Leftrightarrow$ (iii). Since the "$\Leftarrow$" part is obvious, it suffices to show the converse part. Suppose that (ii) is true. Take any $i \in N$. If $i \in \mathcal{S}_P(x_N, R_N)$, then the claim follows from (ii). Suppose that $i \notin \mathcal{S}_P(x_N, R_N)$. Then, we can find $k \in \mathcal{S}_P(x_N, R_N)$ such that $U(x_i, R_i) \cap L(x_k, R_k) = \emptyset$. By (ii), there exists $j \in N$ and $L(x_k, R_k) \supseteq L(y_j, R_j)$. Since $L(x_i, R_i) \supseteq L(y_k, R_k)$ and $L(x_k, R_k) \supseteq L(y_j, R_j)$, we have $L(x_i, R_i) \supseteq L(y_j, R_j)$. This complete the proof. ■

Consider the Pazner-Schmeidler SOF, the social nested-contour SQF, and the pairwise nested-contour SQF. All of the three rules are strongly equitable in the sense that they care only about worst-off agents. The difference is in how they specify the set of worst-off agents. The Pazner-Schmeidler SOF is a refinement of the social nested-contour SQF, and the latter a refinement of the nested-contour SQF (without either of these being extensions, as noted earlier). Note that

$$\left\{i \in N : u_R^\Omega(x_i, R_i) = \min_j u_R^\Omega(x_j, R_j)\right\} \subseteq \mathcal{S}_S(x_N, R_N) \subseteq \mathcal{S}_P(x_N, R_N). \tag{2}$$

A refinement gives us a smaller set of worst-off agents.

In each rule, the worst-off agents are equally treated in the sense that any transfer from some worst-off agent to another cannot be justified. On the other hand, a transfer from a better-off agent to a worst-off agent is always beneficial. Then, the set of worst-off agents must be identical with the set of individuals under any optimal allocation. It is easy to see that if the set of worst-off is small, then there exists a large set of beneficial transfers.

Let us now make a comparison with the Paretian quasi-ordering, which is a foundation of Pareto efficiency. There exists no specification of the worst-off agents under the Paretian quasi-ordering (or it can be interpreted that all individuals are worst-off in any allocations). In other words, any transfer from a person to another person cannot be justified because of the absence of non-worst agents.[8]

[8] The specification of worst-off agents is closely related with the separability of social judgment. In the strong from, separability states that the existence of unconcerned individuals do not matter for social ordering. None of the rules studied in this paper satisfies this property. However, each of them satisfies its weak form. Let $W(x_N, R_N) \subseteq N$ be the mapping that defines the set of worst-off agents for any given rule. Each rule satisfies the following separability property: for any N with $\#N \geq 2$, any $i \notin W(y_N, R_N)$ such that $x_i = y_i$, $x_N R(e) y_N \Leftrightarrow x_{N\setminus\{i\}} R(R_{N\setminus\{i\}}) y_{N\setminus\{i\}}$.

3.3 Convex hull of social upper contour set

In this subsection, we briefly discuss a social quasi-ordering that lies in between the social nested-contour rule and the Egalitarian Walras SOF. The following rule is based on the set-inclusion relation in terms of the convex hull of social upper contour set:[9]

$$x_N R_W(e) y_N \Leftrightarrow co\big(\mathcal{U}(x_N, R_N)\big) \subseteq co\big(\mathcal{U}(y_N, R_N)\big),$$

where $co(A)$ is the convex hull of A. By the monotonic property of co, if $\mathcal{U}(x_N, R_N) \subseteq \mathcal{U}(y_N, R_N)$, then $co\big(\mathcal{U}(x_N, R_N)\big) \subseteq co\big(\mathcal{U}(y_N, R_N)\big)$. Thus, this is a natural refinement of R_S.[10] Remember that R_S is the intersection of all the Pazner-Schmeidler SOFs. Here, R_W is clearly associated with the Egalitarian Walras SOF R^{Ω}_{EW}. Taking the intersection of the Egalitarian Walras SOF R^{Ω}_{EW}, we can obtain R_W (on the subdomain of strictly monotonic preferences).

As in the case of R_P and R_S, we can consider an asymmetric version of R_W, which is useful to consider the relationship among the three rules. Let

$$\mathcal{L}^*(x_N, R_N) = X - \mathrm{Int}\Big(co\big(\mathcal{U}(x_N, R_N)\big)\Big).$$

Define

$$x_N \widehat{R}_W(e) y_N \Leftrightarrow co\big(\mathcal{U}(x_N, R_N)\big) \cap \mathcal{L}^*(y_N, R_N) = \emptyset.$$

$\widehat{R}_W$ is a refinement of $\widehat{R}_S$, which means that it is also a refinement of $\widehat{R}_P$.

3.4 Nietzsche-type social welfare criteria

There is another way to construct SQFs. Remember that $x_N R_P(e) y_N$ if and only if $\forall i \in N, \exists j \in N, L(x_i, R_i) \supseteq L(y_j, R_j)$. Analogically, we can construct the following rule:

$$x_N R_{Niet1}(e) y_N \Leftrightarrow \exists i \in N, \forall j \in N, L(x_i, R_i) \supseteq L(y_j, R_j).$$

Note that it always generates transitive social preferences. Since the formulation of R_{Niet1} is parallel to that of R_P, it seems to be compelling. However, it is not an equitable rule and it is independent of R_P. We can say that the implication of R_{Niet1} is completely opposite to that of R_P. According to R_{Niet1}, it is enough that there exists only one very rich person, and thus a

[9]In this subsection, we emphasize a link between the convex hull of the social upper contour set and the Egalitarian Walras SOF. This quasi-ordering is also associated with the idea of money metrics, which is discussed in the introduction. Consider a criterion maximizing the minimum level of individual money metrics. It depends on a reference price. If we take the intersection of all money-metric maximin criteria with regard to reference prices, we get the convex hull of the social upper contour set.

[10]It is not an extension of R_S because one can have $\mathcal{U}(x_N, R_N) \subsetneq \mathcal{U}(y_N, R_N)$ and $co\big(\mathcal{U}(x_N, R_N)\big) = co\big(\mathcal{U}(y_N, R_N)\big)$.

transfer from the poor to the rich can be socially beneficial.[11]

Let us define the following rule:

$$x_N R_{Niet2}(e) y_N \Leftrightarrow \bigcup_{i \in N} L(x_i, R_i) \supseteq \bigcup_{i \in N} L(y_i, R_i).$$

A dominance relation is constructed in terms of the union of individuals' lower-contour sets. This is a counterpart of R_S. The relationship between R_P and R_S is corresponding to R_{Niet1} and R_{Niet2}. Similarly, the relationship between R_S and R^{Ω}_{PS} is corresponding to R_{Niet1} and the rule, defined as follows:

$$x_N R^{\Omega}_{Niet}(e) y_N \Leftrightarrow \max_{i \in N} u^{\Omega}_R(x_i, R_i) \geq \max_{i \in N} u^{\Omega}_R(y_i, R_i).$$

We can show that by taking the intersection of R^{Ω}_{Niet} over the largest reference set, we can get R_{Niet2}. As a consequence, R^{Ω}_{Niet}, $R_{Niet1}(e)$, and $R_{Niet2}(e)$ can be called Nietzsche-type social welfare criteria. Only the best-off agents matter for social judgments with these criteria.

4 Axiomatic Analysis

In this section, we provide an axiomatic analysis of the two main social quasi-orderings. To do so, we list several basic axioms. First, we introduce Paretian axioms, which basically require that the unanimous agreement must be respected in a certain way. We note that strong Pareto implies weak Pareto and Pareto indifference.

Strong Pareto: For all $e = R_N \in \mathcal{E}$ and $x_N, y_N \in X^N$, if $x_i R_i y_j$ for all $i \in N$, then $x_N R(e) y_N$; if $x_i R_i y_i$ for all $i \in N$ and $x_i P_i y_i$ for some $i \in N$, then $x_N P(e) y_N$

Pareto Indifference: For all $e = R_N \in \mathcal{E}$ and $x_N, y_N \in X^N$, if $x_i I_i y_i$ for all $i \in N$, then $x_N I(e) y_N$.

Weak Pareto: For all $e = R_N \in \mathcal{E}$ and $x_N, y_N \in X^N$, if $x_i P_i y_i$ for all $i \in N$, then $x_N P(e) y_N$.

Hansson independence is an axiom of independence of irrelevant indifference curves. This axiom is a weakening of Arrow's IIA and is proposed by Hansson (1973).[12] Let $I(x_i, R_i) = U(x_i, R_i) \cap L(x_i, R_i) =$.

Hansson independence: For all $e = R_N, e' = R'_N \in \mathcal{E}$ and $x_N, y_N \in X^N$, if for all $i \in N$,

$$I(x_i, R_i) = I(x_i, R'_i) \text{ and } I(y_i, R_i) = I(y_i, R'_i),$$

then $R(e)$ and $R(e')$ agree on $\{x_N, y_N\}$.

[11] An undesirable feature of R_{Niet1} is that it does not satisfy the Pareto principle. Even if everyone is better-off, there might not be an individual who can dominate other people.

[12] See also Fleurbaey, Suzumura, and Tadenuma (2005).

Fleurbaey and Trannoy (2003) show that there is a strong tension between transfer axioms and Paretian axioms. Therefore, a transfer should be restricted to certain situations. The following axiom is a basic transfer axiom in the literature.[13]

Transfer among equals: For all $e = R_N \in \mathcal{E}$ and $x_N, x'_N \in X^N$, if there exist $j, k \in N$ and $\Delta \in \mathbb{R}^\ell_{++}$ such that $R_j = R_k$,

$$x_j = y_j - \Delta > y_k + \Delta = x_k,$$

and $x_i = y_i$ for all $i \in N \setminus \{j, k\}$, then $x_N R(e) y_N$.

We now present the fundamental observation, which is not new; the proof is almost the same as that of Lemma A.1 of Fleurbaey and Maniquet (2011, p 242), and thus, we omit it. The axioms listed in this lemma is imposed in most works in fair social orderings.

Lemma 1. Suppose that an SQF satisfies weak Pareto, transfer among equals, and Hansson independence. For all $e = R_N \in \mathcal{E}$ and $x_N, y_N \in X^N$, if there exist $j, k \in N$ such that

$$y_j P_j x_j, x_k P_k y_k \text{ and } U(x_j, R_j) \cap L(x_k, R_k) = \emptyset,$$

and $x_i P_i y_i$ for all $i \in N \setminus \{j, k\}$, then $x_N P(e) y_N$.

The following result provides an axiomatic characterization of the pairwise nested social quasi-ordering.

Theorem 2.
(i) R_P is an SQF that satisfies weak Pareto, Pareto indifference, transfer among equals, and Hansson independence.
(ii) If an SQF satisfies weak Pareto, transfer among equals, and Hansson independence, then for all $e = R_N \in \mathcal{E}$,

$$x_N \widehat{P}_P(e) y_N \Rightarrow x_N P(e) y_N.$$

Proof. (i) It is clear that R_P satisfies weak Pareto, Pareto indifference, and Hansson independence. It suffices to show that it satisfies transfer among equals. Take $x_N, y_N \in X^N$. Now, suppose that there exist $k, j \in N$ such that $R_k = R_j$ and $\Delta \in \mathbb{R}^\ell_{++}$ such that

$$x_j = y_j - \Delta > y_k + \Delta = x_k,$$

and $x_i = y_i$ for all $i \in N \setminus \{k, j\}$.

One has

$$L(x_i, R_i) \supseteq L(y_i, R_i) \text{ for all } i \neq j. \tag{3}$$

[13] See Fleurbaey and Trannoy (2003) and Fleurbaey and Maniquet (2008, 2011).

Now consider j. Since $U(x_j, R_j) \cap L(y_k, R_k) = \emptyset$, one has $L(x_j, R_j) \supseteq L(y_k, R_k)$. Therefore, transfer among equals is satisfied.

(ii) Suppose that an SQF satisfies weak Pareto, transfer among equals, and Hansson independence. Let $x_N, y_N \in X^N$ and $e = R_N \in \mathcal{R}^N$ be such that

$$\forall i \in N, \exists j \in N, U(x_i, R_i) \cap L(y_j, R_j) = \emptyset. \tag{4}$$

Now, let $M = \{i \in N : x_i P_i y_i\}$. If $M = N$, then we have $x_N P(e) y_N$. In the rest of this proof, we assume that $M \neq N$, i.e., $N \setminus M$ is non-empty.

Take $\#N \setminus M = s$. Without loss of generality, assume that $N \setminus M = \{1, 2, \ldots, s\}$. Consider individual 1. By (4), there exists $j^* \in N$ such that $U(x_k, R_k) \cap L(y_{j^*}, R_{j^*}) = \emptyset$. Define $z_N^1 = (z_1^1, \ldots, z_n^1) \in X^N$ as follows. For a small vector $\varepsilon \in \mathbb{R}^\ell_{++}$, let us consider $e' = R'_N \in \mathcal{R}^N$ such that

$$I(x_i, R_i) = I(x_i, R'_i) \text{ and } I(y_i, R_i) = I(y_i, R'_i) \text{ for all } i \in N,$$

$$\begin{aligned}
&z_1^1 = y_{j^*} + \varepsilon \text{ and } z_{j^*}^1 = y_{j^*} + \frac{\varepsilon}{2},\\
&x_1 P'_1 z_1^1 \text{ and } U(z_1^1, R'_1) \cap L(z_{j^*}^1, R'_{j^*}) = \emptyset,\\
&z_i^1 P'_i y_i \text{ for all } i \in (M)^c \setminus \{1, j^*\},\\
&x_i P'_i z_i^1 P'_i y_i \text{ for all } i \in M \setminus \{j^*\},
\end{aligned}$$

and

$$\mathcal{S}_P(z_N^1, R'_N) = \mathcal{S}_P(y_N, R'_N) \text{ and } [\forall i \in N, \exists j \in N, U(x_i, R'_i) \cap L(z_j^1, R'_j) = \emptyset]. \tag{5}$$

By Lemma 1, we have

$$z_N^1 P(e') y_N.$$

Next, we consider individual 2. Note that

$$\{i \in N : x_i P'_i z_i^1\} = M \cup \{1\} := M_1.$$

Since (5) holds, an argument similar to the previous one works. We can find $k^* \in N$ such that $U(z_2^1, R_2) \cap L(z_{k^*}^1, R_{k^*}) = \emptyset$. By taking an arbitrarily small vector $\varepsilon' \in \mathbb{R}^\ell_{++}$, we can construct $z_N^2 = (z_1^2, \ldots, z_n^2) \in X^N$ and $e'' = R''_N \in \mathcal{R}^N$ as follows:

$$I(x_i, R'_i) = I(x_i, R''_i), I(y_i, R_i) = I(y_i, R'_i), \text{ and } I(z_i^1, R_i) = I(z_i^1, R'_i) \text{ for all } i \in N,$$

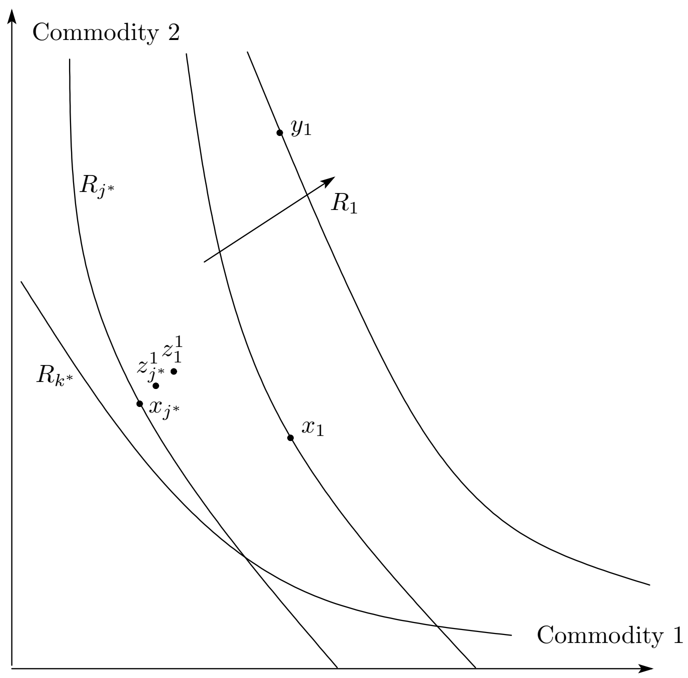

Figure 5: The construction

$$
\begin{aligned}
&z_2^2 = z_{k^*}^1 + \varepsilon' \text{ and } z_{k^*}^2 = z_{k^*}^1 + \frac{\varepsilon'}{2},\\
&x_2 P_2'' z_2^2 \text{ and } U(z_2^2, R_2'') \cap L(z_{k^*}^2, R_{k^*}'') = \emptyset,\\
&z_i^2 P_i'' z_i^1 \text{ for all } i \in (M_1)^c \setminus \{2, k^*\},\\
&x_i P_i'' z_i^2 P_i'' z_i^1 \text{ for all } i \in M_1 \setminus \{k^*\},
\end{aligned}
$$

and

$$\mathcal{S}_P(z_N^2, R_N'') = \mathcal{S}_P(z_N^1, R_N'') \text{ and } [\forall i \in N, \exists j \in N, U(x_i, R_i'') \cap L(z_j^2, R_j'') = \emptyset].$$

By Lemma 1, we have

$$z_N^2 P(e'') z_N^1.$$

Note that

$$\{i \in N : x_i P_i'' z_i^2\} = M \cup \{1, 2\}.$$

Similarly, we can construct a sequence $z^1, z^2, \ldots, z^m$ and obtain a resulting profile $\hat{e} \in \mathcal{R}^N$ such that

$$z^m P(\hat{e}) \ldots z^3 P(\hat{e}) z^2 P(\hat{e}) z^1 R(\hat{e}) y_N,$$

and

$$x_i \hat{P}_i z_i^m \text{ for all } i \in N.$$

By weak Pareto, we have $x_N P(\hat{e}) z_N^m$. By transitivity, we have $x_N P(\hat{e}) y_N$. By Hansson independence, $x_N P(e) y_N$. ■

The theorem implies that if weak Pareto, transfer among equals, and Hansson independence are satisfied, then an SQF is an extension of $\widehat{R}_P$. One might think that if we add some continuity property to weak Pareto, transfer among equals, and Hansson independence, we make an SQF to be a refinement of R_P. The answer to this question is negative. Given some vector Ω, R_{PS}^{Ω} satisfies weak Pareto, transfer among equals, Hansson independence, and continuity, but it is not an extension of R_P. Therefore, a continuity property is not sufficient to obtain a refinement of R_P.

Now, we consider an axiomatic justification of the social nested-contour SQF. To characterize it, we need one more axiom.

The following axiom is a weak equity requirement, saying that making a transfer from a non-advantaged individual to an advantaged one is worsening the allocation. Advantage is defined here in terms of envy. An advantaged individual is one who cannot escape the envy of others without having to make a sacrifice. Formally, i is advantaged in x_N if for all $z R_i x_i$, there is $j \in N$ such that $z P_j x_j$.

Advantage Equity: For all $e = R_N \in \mathcal{E}$, and all $x_N, y_N \in X^N$, if there exist $j, k \in N$ such that j is advantaged in x_N and k is not, and $\Delta \in \mathbb{R}_{++}^{\ell}$ such that

$$y_j = x_j + \Delta,\ y_k = x_k - \Delta,$$

while $y_i = x_i$ for all $i \neq j, k$, then $x_N R(e) y_N$.

It seems that advantage equity is more demanding than transfer among equals in the sense that it applies to individuals with different preferences. However, advantage equity requires nothing about transfers among advantaged individuals, while transfer among equals can be applied to two advantaged individuals. Thus, the two equity axioms are independent.

Lemma 2. Suppose that an SQF satisfies weak Pareto, advantaged equity, and Hansson independence. For all $e = R_N \in \mathcal{E}$ and $x_N, y_N \in X^N$, if there exists an advantaged individual $j \in N$ at x_N such that $y_j P_j x_j$ and $x_i P_i y_i$ for all $i \in N \setminus \{j\}$, then $x_N P(e) y_N$.

Proof. First, consider the case where $x \in U(y_j, R_j)$, $x' \in L(x_j, R_j)$, and $\neg(x > y)$ for some $x, x' \in X$.

Since N is finite, we can always find an individual who is not advantaged. Let $k \in N$ be an individual who is not advantaged at y_N. We now construct a profile and a sequence of allocations.

Let $e' = R'_N$ be such that

$$I(x_i, R'_i) = I(x_i, R_i) \text{ for all } i \in N;$$
$$I(y_i, R'_i) = I(y_i, R_i) \text{ for all } i \in N,$$

and there exist $z_j^1, z_j^2, z_j^3, z_j^4, z_k^1, z_k^2, z_k^3, z_k^4 \in X$ be such that, for some $\Delta \gg 0$,

$$\begin{aligned}
&z_j^1 P'_j y_j;\\
&z_j^2 = z_j^1 - \Delta;\\
&y_j P'_j z_j^2;\\
&z_j^3 P'_j z_j^2;\\
&z_j^4 = z_j^3 - \Delta;\\
&z_j^2 P'_i x_j;\\
&x_j P'_j z_k^4,
\end{aligned}$$

and

$$x_k P'_k z_k^4 P'_k z_k^3 P'_k z_k^2 P'_k z_k^1 P'_k y_k,$$

where $z_k^2 = z_k^1 + \Delta$ and $z_k^4 = z_k^3 + \Delta$, the following is satisfied:

$$U(z_k^4, R'_k) \cap \Big(\Big(\bigcap_{i \neq k,j} L(y_i, R'_i)\Big) \cap L(z_j^4, R'_j)\Big) \neq \emptyset, \tag{6}$$

and j is advantaged at $(z_j^4, x_{N\setminus\{j\}})$. The key of the construction is the choice of $z_j^1, z_j^2, z_j^3, z_j^4$ and indifference curves among these bundles. Figure 6 shows them. Note that (6) makes individual k non-advantaged at $(z_j^4, z_k^4, y_{N\setminus\{k,j\}})$. This is possible because k is not advantaged at y_N. Moreover, we take a sufficiently small bundle $\varepsilon \in \mathbb{R}^\ell_{++}$ such that, for $h = 1, \dots, 4$,

$$z_i^h = y_i + h\varepsilon \text{ and } x_i P'_i z_i^4 \text{ for all } i \in N \setminus \{k, j\}.$$

We here derive the claim by using the constructed profile and sequence. First, weak Pareto implies that

$$z_N^1 R(e') y_N.$$

Second, we make a move from z_N^1 to $(z_k^2, z_j^2, z^1_{N\setminus\{k,j\}})$. Since individual j is advantaged at x_N, j is also advantaged at $(z_k^2, z_j^2, z^1_{N\setminus\{k,j\}})$ because j is better-off and all the other individuals are worse-off by a change from x_N to $(z_k^2, z_j^2, z^1_{N\setminus\{k,j\}})$. Moreover, k is not advantaged at $(z_k^2, z_j^2, z^1_{N\setminus\{k,j\}})$

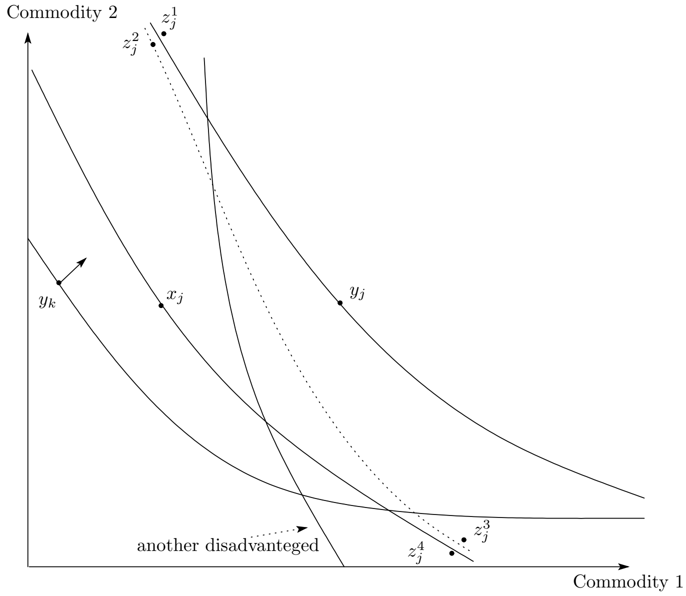

Figure 6: Construction for Lemma 2

because, from (6), we have

$$U(z_k^2, R'_k) \cap \Big(\Big(\bigcap_{i \neq k,j} L(z_i^1, R'_i)\Big) \cap L(z_j^2, R'_j)\Big) \neq \emptyset.$$

By advantage equity, we obtain

$$(z_k^2, z_j^2, z_{N\setminus\{k,j\}}^1) R(e') z_N^1.$$

Third, weak Pareto implies that

$$z_N^3 P(e')(z_k^2, z_j^2, z_{N\setminus\{k,j\}}^1).$$

Fourth, we make a move from z_N^3 to $(z_k^4, z_j^4, z_{N\setminus\{k,j\}}^3)$. Since j is advantaged at $(z_j^4, x_{N\setminus\{j\}})$, j is advantaged at $(z_k^4, z_j^4, z_{N\setminus\{k,j\}}^3)$. Note k is not advantaged by (6). By advantage equity, we obtain

$$(z_k^4, z_j^4, z_{N\setminus\{k,j\}}^3) R(e') z_N^3.$$

Finally, weak Pareto implies that

$$x_N P(e')(z_k^4, z_j^4, z_{N\setminus\{k,j\}}^3).$$

Transitivity implies $x_N P(e') y_N$. By Hansson independence,

$$x_N P(e) y_N \Leftrightarrow x_N P(e') y_N.$$

This completes the proof of this case.

Second, consider the case where $x \in U(y_j, R_j)$, $x' \in L(x_j, R_j)$, and $\neg(x > y)$ for no $x, x' \in X$. This case is proved by an argument similar to the second case in the proof of Lemma A.1 (Fleurbaey and Maniquet, 2011). ■

Theorem 3. (i) R_S is an SQF that satisfies weak Pareto, Pareto indifference, transfer among equals, advantage equity, and Hansson independence.
(ii) If an SQF satisfies weak Pareto, advantage equity, and Hansson independence, then

$$x_N \widehat{P}_S(e) y_N \Rightarrow x_N P(e) y_N.$$

Proof. (i) It is clear that R_S satisfies weak Pareto, Pareto indifference, transfer among equals, Hansson independence, and advantage equity.

(ii) Now, assume that R satisfies weak Pareto, advantage equity, and Hansson independence. Let $x_N, y_N \in X^N$ and $e = R_N \in \mathcal{R}^N$ be such that

$$\mathcal{L}(y_N, R_N) \cap \mathcal{U}(x_N, R_N) = \emptyset.$$

Note that changes outside of indifference curves over x_N and y_N do not affect the ranking between x_N and y_N by Hansson independence.

Let

$$\widehat{\mathcal{S}}(x_N, R_N) = \{i \in N : \mathcal{L}(x_N, R_N) \cap U(x_i, R_i) \neq \emptyset\}.$$

Define

$$M = \{i \in N : x_i P_i y_i\}.$$

If $M = N$, then we have $x_N P(e) y_N$ by weak Pareto. Now, we suppose that $N \setminus M \neq \emptyset$ and $\# N \setminus M = s$. Without loss of generality, we can assume that $N \setminus M = \{1, \ldots, s\}$. For each $i \in N \setminus M$, we have $y_i R_i x_i$.

We now show that each $k \in \{1, \ldots, s\}$ is advantaged at y_N. If $\widehat{\mathcal{S}}(y_N, R_N) \not\subseteq M$, then there exists $i \in N$ such that $\mathcal{L}(y_N, R_N) \cap U(y_i, R_i) \neq \emptyset$ and $y_i \in U(x_i, R_i)$. Since $y_i \in U(x_i, R_i)$, we have $U(y_i, R_i) \subseteq U(x_i, R_i)$, which implies that $U(y_i, R_i) \subseteq \mathcal{U}(x_N, R_N)$. Now, $\mathcal{L}(y_N, R_N) \cap U(y_i, R_i) \neq \emptyset$ and $U(y_i, R_i) \subseteq \mathcal{U}(x_N, R_N)$ implies that $\mathcal{L}(y_N, R_N) \cap \mathcal{U}(x_N, R_N) \neq \emptyset$. This is a contradiction. Therefore, we have the following:

$$\widehat{\mathcal{S}}(y_N, R_N) \subseteq M. \tag{7}$$

Take some individual $k \in \{1, \ldots, s\}$. Since $k \in N \setminus M$, it follows that $k \notin \widehat{\mathcal{S}}(y_N, R_N)$ by (7). This means that

$$\mathcal{L}(y_N, R_N) \cap U(y_k, R_k) = \emptyset.$$

Thus, k is advantaged at y_N.

Now, we construct a profile and a sequence of allocations. By taking a very small bandle $\varepsilon \in \mathbb{R}^\ell_{++}$ and a smaller one $\delta \ll \varepsilon/\#N$, let $e' = R'_N \in \mathcal{R}^N$ and $z^1_N, z^2_N, \ldots, z^s_N$ be such that

$$I(x_i, R'_i) = I(x_i, R_i) \text{ for all } i \in N;$$
$$I(y_i, R'_i) = I(y_i, R_i) \text{ for all } i \in N,$$

and, for each $k \in \{1, \ldots, s\}$,

$$\begin{aligned}
z^k_1 &= x_1 - \varepsilon + (k-1)\delta;\\
z^k_2 &= x_2 - \varepsilon + (k-2)\delta;\\
&\vdots\\
z^k_j &= x_j + \varepsilon + (k-j)\delta;\\
&\vdots\\
z^k_{k-1} &= x_{k-1} - \varepsilon + \delta;\\
z^k_k &= x_k - \varepsilon;\\
z^k_{k+1} &= y_{k+1} + k\varepsilon;\\
&\vdots\\
z^k_n &= y_n + k\varepsilon
\end{aligned}$$

,

and

$$\widehat{S}(z^k_N; R_N) = \widehat{S}(y_N; R_N) \text{ for all } k \in \{1, \ldots, s\};$$
$$x_i P_i z^s_i \text{ for all } i \in N.$$

Figure 7 demonstrates the construction of these allocations (we assume that $N = \{1, 2, 3\}$ and $s = 2$). The key is that advantaged individuals at y_N are also advantaged at $z^1_N, z^2_N, \ldots, z^s_N$. Taking a sufficiently small ε makes this construction possible.

Now, we prove the claim by employing the constructed profile and sequence. First, consider a move to z^1_N from y_N. Note that

$$z^1_1 = x_1 - \varepsilon \text{ and } z^1_i = y_i + \varepsilon \text{ for all } i \neq 1.$$

Note that $y_1 P_1 z^1_1$ and $z^1_i P_i y_i$ for all $i \neq 1$. Since $\widehat{S}(z^1_N; R_N) = \widehat{S}(y_N; R_N)$, individual 1 is advantaged

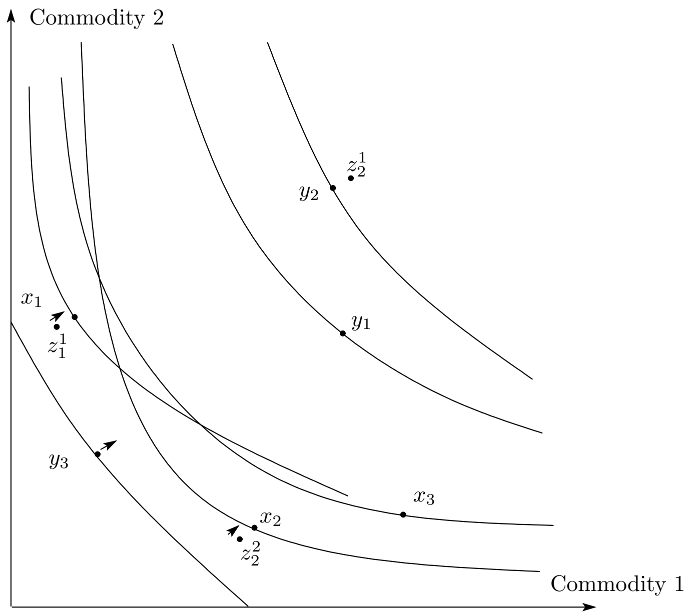

Figure 7: Construction for Theorem 3

at z_N^1. By Lemma 2, we have

$$z_N^1 P(e') y_N.$$

Next, consider a move to z_N^2 from z_N^1. By construction,

$$z_1^2 = x_1 - \varepsilon + \delta, z_2^2 = x_2 - \varepsilon, \text{ and } z_i^1 = y_i + 2\varepsilon \text{ for all } i \neq 1, 2.$$

Note that $z_2^1 P_1 z_2^2$ and $z_i^2 P_i z_i^1$ for all $i \neq 2$. Since $\widehat{S}(\hat{z}_N^2; R_N) = \widehat{S}(y_N; R_N)$, individual 2 is advantaged at z_N^2. By Lemma 2, we have

$$z_N^2 P(e') z_N^1.$$

By repeating this process, we have

$$z_N^s P(e') \ldots P(e') z_N^3 P(e') z_N^2 P(e') z_N^1 P(e') y_N,$$

Weak Pareto implies that

$$x_N P(e') z_N^s$$

Transitivity implies that $x_N P(e') y_N$. By Hansson independence,

$$x_N P(e) y_N \Leftrightarrow x_N P(e') y_N.$$

This completes the proof. ■

As in the case of R_P, a continuity property is not sufficient to obtain an extension of R_S.

5 Conclusion

We conclude this paper with some remarks. In social choice, there are two possible forms of incompleteness in the social judgment. Incompleteness may either lie in the measurement of individual wellbeing, as in this paper, or in the aggregation over the population, allowing for different degrees of inequality aversion.

Our SQFs are of the maximin kind, giving absolute priority to the worst-off. Their incompleteness does not come from the aggregation, but only from the well-being measurement.[14] This is true since the combination of Pareto axioms with either transfer among equals or advantage equity pushes for giving absolute priority to the worst-off (Lemmas 1 and 2). To make room for the other form of incompleteness while remaining Paretian, one would need to weaken these equity axioms.

An important step in our SQFs is identifying the set of worst-off individuals. This is a fundamental issue in a multidimensional poverty measurement. In the case of the one-dimensional attribute (typically, income), given a poverty line, one can determine the set of poor individuals without any doubt. However, if there are multiple attributes, such as health, housing, or education, there are various ways to construct the set of poor individuals (Tsui 2002, Atkinson 2003). Suppose that there is a threshold for each attribute. One extreme way is identifying the poor as individuals with no attributes above the threshold. The other extreme way is identifying the poor as individuals with some attributes below the threshold. Counting the number of attributes below the threshold is also a popular approach (Alkire and Foster, 2011). However, most works in this line do not use information about individual preferences, an exception being Decancq et al. (2015b), which relies on fair social orderings. Our approach suggests a new way to use information about the indifference curves of individuals for anti-poverty policy. Either of our two proposed definitions could be used to identify the worst-off.[15] Then, the worst-off can be removed from the dataset, and the worst-off among the remaining population could again be identified, and so on. In this way, a hierarchical identification of the most disadvantaged populations is possible.

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[14] In this paper, we consider only maximin criteria. We can construct quasi-orderings due to incompleteness from well-being measurement for other aggregation rules. For instance, consider the sum of values $u_R^{\Omega}(x_i, R_i)$ of the ray utility function. We can consider a social ordering for each Ω. Then, we can take the intersection of those orderings and get a social quasi-ordering. This is a utilitarian counterpart of the Social Nested-Contour SQF.

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