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Abstract. Consider a singular historic event such as a revolution that triggers social change. This typically involves a rearrangement of the relative social standing accorded to different groups in society. The purpose of a measure of sustained social mobility is to assess the extent to which such changes in the social hierarchy persist over time. We use three intuitively appealing axioms to characterize a measure of sustained social mobility that is based on the well-established Kemeny distance for linear orderings. There is a close family resemblance to Kemeny's rule for aggregating linear individual orderings and, as a consequence, our result also provides an argument in favor of this social welfare function.

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Keywords: rank, sustained social mobility, Kemeny distance, revolution, social class

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1 Introduction

The measurement of mobility has a long-standing tradition in economics and political philosophy. While most contributions focus on changes in income or wealth distributions (see, for instance, Fields and Ok, 1999, for a survey), the mobility inherent in an evolving society with respect to the relative ranking of different groups in the population are of significant importance as well; see, for example, D'Agostino and Dardanoni (2009), Bossert, Can, and D'Ambrosio (2018), and Ghiselli Ricci (2019) for axiomatic analyses. In addition, empirical studies focusing on relative income ranks instead of intergenerational elasticities of income (which had been the main focus for many years) have been attracting attention; see Chetty, Hendren, Kline, and Saez (2014), for example. These studies show that using ranks yields considerably different results than using income values.

Mobility typically examines the transition in a variable such as income or rank from an initial time period to the next. In other words, only two periods are considered. The type of mobility considered in this paper is different because it applies to an initial period and more than a single succeeding period. This is appropriate in scenarios where the long-term effect of a singular event is to be analyzed, standard examples being revolutions such as the French Revolution in the late 18^{th} century and the Meiji Restoration that took place in Japan during the second half of the 19^{th} century. When determining the impact of events of this nature, any change in societal ranks that is merely transitory reflects considerably less social mobility than a change that is sustained over time, and it is this phenomenon that we intend to capture. In a setup where there are a finite number of periods that follow the initial period—period zero—in which the event in question takes place, a natural measure of sustained social mobility consists of the sum of the Kemeny distances (see Kemeny, 1959, and Kemeny and Snell, 1962). Specifically, the Kemeny distance between the ranking in the initial period and each of the subsequent periods is calculated, and adding these values yields the measure of sustained social mobility that we propose. We use three intuitively appealing axioms to characterize this measure of sustained social mobility—the first is a monotonicity condition, the second a complementarity axiom, and the third a normalization property. The measure coincides with the objective function to be minimized according to the Kemeny rule applied in social-aggregation problems.

Our axiomatization applies to linear orderings only; that is, we assume that the orderings in question are antisymmetric. While the restriction to linear orderings employed in this paper certainly involves some loss of generality, the linearity assumption is quite commonly employed in the literature, as evidenced by contributions such as those of Slater (1961), Young and Levenglick (1978), Saari and Merlin (2000), and Kawada (2018) in the context of aggregation rules. There are some features that prevent our result from being easily generalizable to the case of weak (that is, not necessarily linear) orderings. In a sense, this observation is not too surprising and seems to account for the prevalence of the antisymmetry assumption in the social-aggregation framework.

In Section 2, we present the definition of the Kemeny distance and its structural features that are relevant for our purposes. Section 3 provides the definition of our measure of sustained social mobility and its characterization. In Section 4, we explain why it is difficult to generalize our result to weak orderings.

2 The Kemeny distance

Consider a finite set X of $n \geq 2$ groups in a society. A linear ordering on X is a reflexive, complete, transitive, and antisymmetric binary relation $R \subseteq X \times X$. The interpretation is that this linear ordering represents the social positions of the groups. The set of all linear orderings on X is denoted by $\mathcal{R}$. For a linear ordering $R \in \mathcal{R}$, the inverse of R is denoted by R^{-1}.

The Kemeny distance $d_K \colon \mathcal{R}^2 \to \mathbb{R}_+$ between two linear orderings is defined as

$$d_K(R, R') = |R \setminus R'|$$

for all $R, R' \in \mathcal{R}$. A linear ordering R' can be obtained from a linear ordering R by means of an elementary change if the only difference between the two linear orderings is that the positions of two adjacent groups in R are exchanged. The Kemeny distance between any two linear orderings R and R' is equal to the minimal number of elementary changes required to move from R to R'. As a consequence, the range $d_K(\mathcal{R}^2)$ of d_K consists of a finite set of non-negative integers.

By definition, the maximal value of the Kemeny distance is attained for any linear ordering R and its inverse R^{-1}, and it is given by $d_K^{\max} = d_K(R, R^{-1}) = (n-1)n/2$; note that $(n-1)n/2$ is the minimal number of elementary changes required to move from an arbitrary linear ordering to its inverse.

Moreover, it follows that, for any two linear orderings R and R',

$$d_K(R, R') + d_K(R, R'^{-1}) = d_K^{\max} = \frac{(n-1)n}{2}; \tag{1}$$

this is the case because any linear ordering R must contain $(n-1)n/2$ pairs with distinct elements and each of these elements must be either in R' or in R'^{-1}.

The Kemeny distance is arguably the most commonly-employed distance function for linear orderings; its generalization to orderings that are not necessarily antisymmetric has a similarly privileged standing. See, for instance, Kemeny (1959), Kemeny and Snell (1962), and Can and Storcken (2018) for characterizations of the Kemeny distance.

3 Sustained social mobility

There is an initial period 0 in which an event (such as a revolution) takes place, and there are $T \geq 2$ subsequent periods. A measure of sustained social mobility is a function $M \colon \mathcal{R}^{T+1} \to \mathbb{R}_+$. The interpretation of $M(R_0, R_1, \ldots, R_T)$ is that this value measures the extent to which the event increased social mobility in later time periods—that is, the measure indicates the persistence of the social change triggered by the event. We propose to use the Kemeny measure M_K of sustained social mobility, defined as

$$M_K(R_0, R_1, \ldots, R_T) = \sum_{t=1}^{T} d_K(R_0, R_t)$$

for all $(R_0, R_1, \ldots, R_T) \in \mathcal{R}^{T+1}$. Note that $M_K(R_0, R_1, \ldots, R_T) = 0$ if and only if $R_t = R_0$ for all $t \in \{1, \ldots, T\}$ and that $\max_{(R_0,R_1,\ldots,R_T)\in\mathcal{R}^{T+1}} M_K((R_0, R_1, \ldots, R_T)) = M_K((R_0, R_0^{-1}, \ldots, R_0^{-1})) = T d_K^{\max}$. According to M_K, there is no sustained social mobility if there does not occur any change in the ranking of the groups, and the largest sustained social mobility is obtained if the ranking of the groups in the initial period is reversed in the next period and this reversal persists in all subsequent periods.

If the linear orderings $R_1, \ldots, R_T$ are interpreted as the goodness relations of the members of a society defined on a set of alternatives, M_K is nothing but the objective function to be minimized in the definition of Kemeny's aggregation rule; see, for example, Kemeny (1959), Slater (1961), Young and Levenglick (1978), and Saari and Merlin (2000), to name but a few. Young and Levenglick (1978) and Young (1988) link Kemeny's rule to a proposal by Condorcet (1785). They also argue that median-based variants of the Kemeny rule can be considered to be superior to those that are mean-based. Kawada (2018) shows that the Borda (1781) rule is equivalent to the rule that maximizes the cosine similarity measure between a linear social ordering and a profile of linear individual orderings. The cosine measure appears in the computer-science literature.

To illustrate how our measure of sustained social mobility evaluates the long-term effect of a singular event, assume that there are three groups (classes) in a society and that $T = 3$. Thus, there are four generations, one alive in each period. In period 0, the aristocrats a are the highest-ranked class, the capitalists c are the second-highest, and the workers w are the lowest-ranked. There is a revolution that takes place in period 0, which leads to a complete rank reversal in the first period. However, as is often the case, the aristocrats gradually come back to the top. This situation is illustrated by means of the linear orderings that apply in each period, that is,

$$aR_0cR_0w;$$
$$wR_1cR_1a;$$
$$cR_2wR_2a;$$
$$aR_3wR_3c.$$

We obtain $d_K(R_0, R_1) = 3$, $d_K(R_0, R_2) = 2$, and $d_K(R_0, R_1) = 1$ so that the corresponding value of our measure of sustained social mobility is given by

$$M_K(R_0, R_1, R_1, R_3) = 6.$$

We employ three axioms in our characterization of M_K. The first of these is a plausible monotonicity property. It states that a ceteris-paribus move of a relation in one of the periods 1 to T closer to the initial relation in period 0 reduces sustained social mobility.

Monotonicity. For all $R_0 \in \mathcal{R}$, for all $(R_1, \ldots, R_T), (R_1', \ldots, R_T') \in \mathcal{R}^T$, and for all $\tau \in \{1, \ldots, T\}$ such that $R_t' = R_t$ for all $t \in \{1, \ldots, T\} \setminus \{\tau\}$, if $d_K(R_0, R_\tau') < d_K(R_0, R_\tau)$, then

$$M(R_0, R_1', \ldots, R_T') < M(R_0, R_1, \ldots, R_T).$$

The second axiom is a complementarity condition. It requires that the sum of the sustained-mobility values for a given initial ordering R_0 is maximal (in the sense of T

times the maximal Kemeny distance) whenever each of the later orderings R_t is paired with its inverse R_t^{-1} for all $t \in \{1, \ldots, T\}$.

Complementarity. For all $(R_0, R_1, \ldots, R_T) \in \mathcal{R}^{T+1}$,

$$M(R_0, R_1, \ldots, R_T) + M(R_0, R_1^{-1}, \ldots, R_T^{-1}) = T d_K^{\max}.$$

The formulations of these two axioms rely on the Kemeny distance, which may be seen as a shortcoming of our approach. However, that this is not unusual at all is evidenced by the fact that this observation also applies to the characterizations of the Kemeny distance mentioned above. The betweenness condition employed by both Kemeny (1959) and Can and Storcken (2018) also relies on the Kemeny distance itself because it is phrased in terms of the number of elementary changes—which underlies the definition of this distance measure.

Our last axiom is a normalization condition for a measure of sustained social mobility. It postulates that the minimal difference in sustained social mobility between two distinct linear orderings cannot be less than one. The corresponding axiom for a distance function appears in Kemeny (1959) and in Can and Storcken (2018) who characterize the Kemeny distance using it.

Normalization. For all $R_0 \in \mathcal{R}$ and for all $(R_1, \ldots, R_T), (R'_1, \ldots, R'_T) \in \mathcal{R}^T$ with $(R_1, \ldots, R_T) \neq (R'_1, \ldots, R'_T)$,

$$|M(R_0, R_1, \ldots, R_T) - M(R_0, R'_1, \ldots, R'_T)| \geq 1.$$

As a preliminary observation, we show that if a measure of sustained social mobility satisfies monotonicity, complementarity, and normalization, it must assume the value zero whenever all subsequent linear orderings $R_1, \ldots, R_T$ are equal to the initial linear ordering R_0.

Lemma 1. *If a measure of sustained social mobility M satisfies monotonicity, complementarity, and normalization, then $M(R_0, R_0, \ldots, R_0) = 0$ for all $R_0 \in \mathcal{R}$.*

Proof. Let $R_0 \in \mathcal{R}$. Substituting $R_t = R_0$ for all $t \in \{1, \ldots, T\}$ in the definition of complementarity, it follows that

$$M(R_0, R_0, \ldots, R_0) + M(R_0, R_0^{-1}, \ldots, R_0^{-1}) = T d_K^{\max} = T \frac{(n-1)n}{2}. \tag{2}$$

The ordering R_0 can be reached from its inverse R_0^{-1} by means of $(n-1)n/2$ elementary changes. Thus, the $(T+1)$-tuple $(R_0, R_0, \ldots, R_0)$ can be reached from $(R_0, R_0^{-1}, \ldots, R_0^{-1})$ by means of $T(n-1)n/2$ elementary changes. By monotonicity, each of these $T(n-1)n/2$ elementary changes reduces the value of M. Because M satisfies normalization, each of these $T(n-1)n/2$ elementary changes reduces the value of M by at least one. Thus, it follows that

$$M(R_0, R_0, \ldots, R_0) \leq M(R_0, R_0^{-1}, \ldots, R_0^{-1}) - T\frac{(n-1)n}{2}.$$

By (2), it follows that

$$M(R_0, R_0^{-1}, \ldots, R_0^{-1}) - T\frac{(n-1)n}{2} = -M(R_0, R_0, \ldots, R_0)$$

and, therefore,

$$M(R_0, R_0, \ldots, R_0) \leq -M(R_0, R_0, \ldots, R_0)$$

so that $2M(R_0, R_0, \ldots, R_0) \leq 0$ and hence $M(R_0, R_0, \ldots, R_0) \leq 0$. Because the codomain of M is the set of non-negative real numbers, this can only be true if $M(R_0, R_0, \ldots, R_0) = 0$, which was to be established. ■

We now obtain the following characterization of M_K. As the theorem shows, if we require a sustained social mobility measure to be consistent with the use of the Kemeny distance in the sense of monotonicity and complementarity, only the measure M_K is permissible once we also impose normalization.

Theorem 1. *A measure of sustained social mobility M satisfies monotonicity, complementarity, and normalization if and only if $M = M_K$.*

Proof. That M_K satisfies monotonicity, complementarity, and normalization is straightforward to verify. Conversely, suppose that M is a measure of sustained social mobility that satisfies the three axioms. Without loss of generality, suppose that

$$\begin{aligned} M(R_0, R_1, \ldots, R_T) &= M_K(R_0, R_1, \ldots, R_T) + \Delta(R_0, R_1, \ldots, R_T) \\ &= \sum_{t=1}^{T} d_K(R_0, R_t) + \Delta(R_0, R_1, \ldots, R_T) \end{aligned} \quad (3)$$

for all $(R_0, R_1, \ldots, R_T) \in \mathcal{R}^{T+1}$, where $\Delta\colon \mathcal{R}^{T+1} \to \mathbb{R}$ is a function that yields a residual of the values of M and M_K. Let $(R_0, R_1, \ldots, R_T)$ be an arbitrary element of $\mathcal{R}^{T+1}$.

If $R_t = R_0$ for all $t \in \{1, \ldots, T\}$, Lemma 1 immediately implies that

$$\Delta(R_0, R_1, \ldots, R_T) = 0$$

and, therefore,

$$M(R_0, R_1, \ldots, R_T) = M_K(R_0, R_1, \ldots, R_T).$$

Now suppose that there exists $\tau \in \{1, \ldots, T\}$ such that $R_\tau \neq R_0$. By applying $\sum_{t=1}^{T} d_K(R_0, R_t)$ elementary changes, we can obtain the $(T+1)$-tuple of linear orderings $(R_0, R_0, \ldots, R_0)$ from $(R_0, R_1, \ldots, R_T)$ and, for each elementary change, the value of M decreases as a consequence of the monotonicity axiom. This implies

$$0 = M(R_0, R_0, \ldots, R_0) < M(R_0, R_1, \ldots, R_T) = \sum_{t=1}^{T} d_K(R_0, R_t) + \Delta(R_0, R_1, \ldots, R_T)$$

where the first equality follows from Lemma 1. By normalization, the minimal amount by which the value of M decreases with each elementary change is given by one. Because there are $\sum_{t=1}^{T} d_K(R_0, R_t)$ elementary changes required, it follows that

$$0 \leq \sum_{t=1}^{T} d_K(R_0, R_t) + \Delta(R_0, R_1, \ldots, R_T) - \sum_{t=1}^{T} d_K(R_0, R_t)$$

and, as a consequence,

$$\Delta(R_0, R_1, \ldots, R_T) \geq 0. \tag{4}$$

By complementarity,

$$M(R_0, R_1, \ldots, R_T) + M(R_0, R_1^{-1}, \ldots, R_T^{-1}) = T d_K^{\max} = T(n-1)n/2$$

and, therefore,

$$\sum_{t=1}^{T} d_K(R_0, R_t) + \Delta(R_0, R_1, \ldots, R_T) + \sum_{t=1}^{T} d_K(R_0, R_t^{-1}) + \Delta(R_0, R_1^{-1}, \ldots, R_T^{-1}) = T(n-1)n/2.$$

This can be rewritten as

$$\sum_{t=1}^{T} \left[d_K(R_0, R_t) + d_K(R_0, R_t^{-1}) \right] + \Delta(R_0, R_1, \ldots, R_T) + \Delta(R_0, R_1^{-1}, \ldots, R_T^{-1}) = T(n-1)n/2$$

and, using (1), it follows that

$$T(n-1)n/2 + \Delta(R_0, R_1, \ldots, R_T) + \Delta(R_0, R_1^{-1}, \ldots, R_T^{-1}) = T(n-1)n/2$$

which implies

$$\Delta(R_0, R_1, \ldots, R_T) + \Delta(R_0, R_1^{-1}, \ldots, R_T^{-1}) = 0.$$

Because $\Delta(R_0, R_1, \ldots, R_T)$ and $\Delta(R_0, R_1^{-1}, \ldots, R_T^{-1})$ are non-negative by (4), it follows that both of them must be equal to zero and, therefore, (3) implies

$$M(R_0, R_1, \ldots, R_T) = M_K(R_0, R_1, \ldots, R_T). \blacksquare$$

As is evident from the proof of Theorem 1, the complementarity axiom plays a key role in deriving a sustained social mobility measure that possesses a linear structure. However, as we demonstrate in the context of proving the independence of our axioms, the axiom by itself does not imply linearity.

The three axioms of Theorem 1 are independent. The measure of sustained social mobility given by $2M_K$ satisfies monotonicity and normalization and violates complementarity, and the measure defined by

$$M(R_0, R_1, \ldots, R_T) = M_K(R_0, R_1^{-1}, \ldots, R_T^{-1}) = T d_K^{\max} - M_K(R_0, R_1, \ldots, R_T)$$

for all $(R_0, R_1, \ldots, R_T) \in \mathcal{R}^{T+1}$ violates monotonicity and satisfies complementarity and normalization.

To show that normalization is not implied by the other axioms, let $g\colon [0, d_K^{\max}] \to [0, d_K^{\max}]$ be an increasing transformation that satisfies

$$g(0) \geq 0 \tag{5}$$

and

$$g(d_K^{\max} - y) = d_K^{\max} - g(y) \text{ for all } y \in [0, d_K^{\max}]. \tag{6}$$

Now define the measure M by

$$M(R_0, R_1, \ldots, R_T) = \sum_{t=1}^{T} g(d_K(R_0, R_t))$$

for all $(R_0, R_1, \ldots, R_T) \in \mathcal{R}^{T+1}$. Clearly, M satisfies monotonicity and complementarity and violates normalization, provided that a suitable non-linear transformation g such as that of Figure 1 is chosen. Condition (5) guarantees that $M(\mathcal{R}^{T+1}) \subseteq \mathbb{R}_+$, and (6) ensures that the graph of g is symmetric with respect to the point $(d_K^{\max}/2, g(d_K^{\max}/2)) = (d_K^{\max}/2, d_K^{\max}/2)$. An example of a transformation g that satisfies these conditions is illustrated in Figure 1. This example also shows that complementarity by itself is not sufficient to imply that a measure of sustained social mobility is linear.

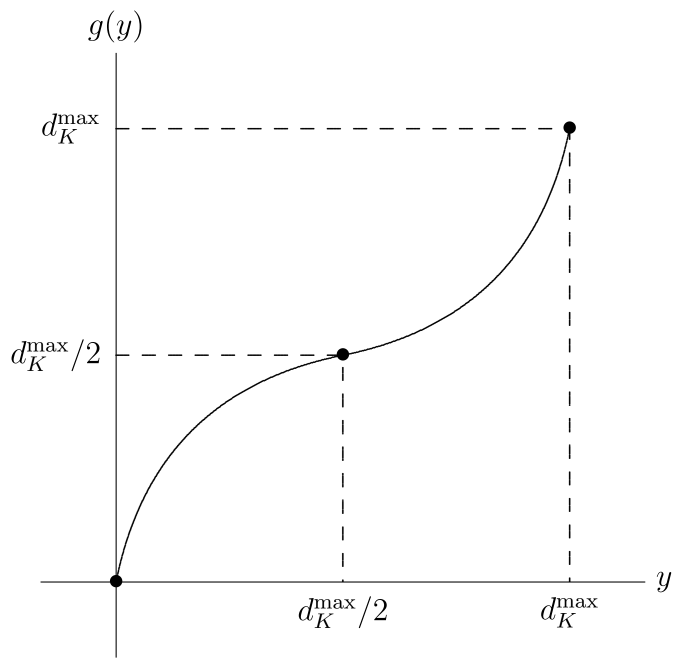

Figure 1: An increasing transformation g that satisfies (5) and (6).

4 Discussion

A straightforward generalization of our characterization result to all (that is, not necessarily antisymmetric) orderings seems elusive. This can, to a large extent, be explained by some peculiarities that emerge when tools such as the Kemeny distance are applied in the more general setting. As illustrated below, the complementarity axiom fails to generalize to the case of weak orderings so that there does not seem to be any natural extension of our result to orderings that are not necessarily antisymmetric.

If two social classes are permitted to have equal status, it is no longer the case that a single elementary change is sufficient to move from one ordering to another. For example, suppose that $X = \{x, y, z\}$ and consider the ordering R given by xPy, xPz, yIz, where P and I are the asymmetric part and the symmetric part of R. In order to move to the adjacent ordering R' given by $xI'y$, $xI'z$, $yI'z$, two elementary changes are required—namely, xPy and xPz have to be changed to $xI'y$ and $xI'z$. The Kemeny distance for orderings R and R' on X is defined as

$$d_K(R, R') = |R \setminus R'| + |R' \setminus R|.$$

According to the Kemeny distance defined in this manner, the maximal Kemeny distance no longer is achieved for all pairs of an ordering and its inverse. The maximal Kemeny distance in the case of three social classes is six, and it is achieved for any pair of a linear ordering and its inverse. However, for orderings that are not linear, this is not the case. For example, the distance between the ordering R given by xPy, xPz, yIz and its inverse R^{-1} defined by $yP^{-1}x$, $zP^{-1}x$, $yI^{-1}z$ is equal to four rather than six. As a consequence, the complementarity axiom no longer is well-defined and, moreover, it is now possible that an ordering is not on a shortest path between a relation and its inverse. For example, with the relation R as just defined and R'' given by $xP''y$, $xP''z$, $yP''z$, it follows that the Kemeny distance between R'' and R is equal to one and the Kemeny distance between R'' and R^{-1} is equal to five so that these two distances add up to six rather than the distance between R and R^{-1}, which is equal to four.

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