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Independent, neutral, and monotonic collective choice: The role of Suzumura consistency*

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Abstract. We examine the impact of Suzumura's (1976) consistency property when applied in the context of collective choice rules that are independent of irrelevant alternatives, neutral, and monotonic. An earlier contribution by Blau and Deb (1977) establishes the existence of a vetoer if the social relation is required to be complete and acyclical. The purpose of this paper is to explore the possibilities that result if completeness and acyclicity are dropped and Suzumura consistency is imposed instead. A conceptually similar but logically independent version of the combined axiom that requires the collective decision mechanism to be independent, neutral, and monotonic is employed. In the case of a finite population, we obtain an alternative impossibility theorem if a collective choice rule is assumed to be non-degenerate and a strong non-dictatorship requirement is imposed. If the population is countably infinite, the impossibility can be avoided but it resurfaces if strong non-dictatorship is extended to a coalitional variant. *Journal of Economic Literature* Classification Nos.: C02, D71.

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1 Introduction

One possible response to Arrow's (1951; 1963; 2012) fundamental impossibility theorem consists of weakening the requirements imposed on a social relation. An early contribution in that vein is Sen's (1969, 1970) characterization of the Pareto extension rule. Sen's result is made possible by weakening transitivity to quasi-transitivity—that is, the requirement that the asymmetric part of the social relation be transitive but not necessarily the relation itself. As is the case for Arrow's framework, completeness is assumed by Sen. An alternative path is followed by Weymark (1984) who drops completeness from Arrow's list of properties but retains transitivity. This leads to a characterization of the Pareto rule. Bossert and Suzumura (2008) use neither completeness nor (quasi-)transitivity but employ Suzumura consistency instead. Suzumura consistency, introduced by Suzumura (1976), is intermediate in strength between transitivity and acyclicity, requiring a preference relation not to exhibit the type of cycle that leads to a money pump. If there are at least as many alternatives as there are individuals, an alternative characterization of the Pareto rule results from Bossert and Suzumura's (2008) characterization. However, if there are fewer alternatives than individuals, the corresponding class of collective choice rules is considerably more rich. The latter case is especially relevant in political elections in which there are vastly more voters than candidates.

Acyclical collective choice is examined, for example, by Brown (1974, 1975), Blau and Deb (1977), Banks (1995), and Bossert and Cato (2020). An important observation by Blau and Deb (1977) shows that if a collective choice rule generates complete and acyclical social relations and satisfies a property that combines the well-established requirements of independence of irrelevant alternatives, neutrality, and monotonicity, then there must exist a vetoer—that is, an individual who can prevent any alternative to be socially better than another alternative by declaring the latter to be individually better for him or her than the former.

This paper complements some of the earlier work alluded to above. In particular, we examine the consequences of removing completeness and strengthening acyclicity to Suzumura consistency in the setting of Blau and Deb (1977). As explained in more detail once its formal definition has been introduced, Suzumura consistency is an important property in that it provides, for example, a previously missing link between necessary and sufficient conditions for the existence of an ordering extension; see Szpilrajn (1930) and Suzumura (1976, 1983). Moreover, unlike the properties of quasi-transitivity and acyclicity, it has a well-defined closure operation. Suzumura consistency also coincides with transitivity in the presence of completeness and can thus be considered a natural weakening of this fundamental condition.

We consider both the case of a finite population and environments in which the population is countably infinite. If there are finitely many individuals, an impossibility is established. The result bears a family resemblance to the vetoer theorem of Blau and Deb (1977) but it does not follow from this earlier observation; in fact, our proof technique differs substantially from that employed by Blau and Deb. We use a non-null property that rules out degenerate cases, a(n independent) variant of the combined independence, neutrality, and monotonicity axiom of Blau and Deb (1977), and a strengthening of Arrow's (1951; 1963; 2012) non-dictatorship requirement. Our property of strong non-dictatorship is equivalent to Arrow's original axiom in the presence of completeness but, without this assumption, it is stronger; notably, it is not compatible with the Pareto rule if social relations are not necessarily complete. As is the case for Blau and Deb's theorem, the assumption that there be at least as many alternatives as individuals is needed. In the infinite population framework, the impossibility can be avoided and a more demanding system of axioms can be accommodated. In particular, weak Pareto rather than merely non-null is satisfied by our existence example, and both versions of independence, neutrality, and monotonicity are among the list of properties. An impossibility emerges if strong non-dictatorship is extended to a coalitional version of the axiom.

Section 2 and 3 introduce binary relations and collective choice rules, along with some of their properties. Section 4 is devoted to the case of a finite population, and countably infinite populations are considered in Section 5. Section 6 concludes.

2 Binary relations

Let X be a non-empty set of alternatives which may be finite or countably infinite with $|X| \geq 3$, and suppose that $R \subseteq X \times X$ is a (binary) relation on X. The set of all relations on X is denoted by $\mathcal{B}$. The symmetric part of R is defined by

$$I(R) = \{(x, y) \in X \times X \mid (x, y) \in R \text{ and } (y, x) \in R\},$$

and the asymmetric part of R is

$$P(R) = \{(x, y) \in X \times X \mid (x, y) \in R \text{ and } (y, x) \notin R\}.$$

For any two alternatives $x, y \in X$ such that $x \neq y$, the restriction of a relation R to $\{x, y\}$ is denoted by $R|_{\{x,y\}}$.

The relation R is complete if, for all $x, y \in X$,

$$(x, y) \in R \text{ or } (y, x) \in R.$$

Completeness is sometimes restricted to pairs $(x, y) \in X \times X$ such that x and y are distinct, and the case in which the two alternatives are identical is stated as the separate axiom of reflexivity. Because this distinction is not of relevance for the purposes of this paper, we use a single property for simplicity.

The standard coherence requirement imposed on a relation is that of transitivity. A relation R is transitive if, for all $x, y, z \in X$,

$$(x, y) \in R \text{ and } (y, z) \in R \;\Rightarrow\; (x, z) \in R.$$

A complete and transitive binary relation is called an ordering, and the set of all orderings on X is denoted by $\mathcal{R}$.

Two commonly employed weakenings of transitivity are quasi-transitivity and acyclicity. The relation R is quasi-transitive if the asymmetric part $P(R)$ of R is transitive, and R is acyclical if, for all $K \in \mathbb{N}$ and for all $x^0, \ldots, x^K \in X$,

$$(x^{k-1}, x^k) \in P(R) \text{ for all } k \in \{1, \ldots, K\} \;\Rightarrow\; (x^K, x^0) \notin P(R).$$

An important strengthening of acyclicity is introduced by Suzumura (1976) under the name of consistency. To avoid confusion with other (unrelated) uses of this label in the literature, we refer to the axiom as Suzumura consistency. A relation R is Suzumura consistent if, for all $K \in \mathbb{N}$ and for all $x^0, \ldots, x^K \in X$,

$$(x^{k-1}, x^k) \in R \text{ for all } k \in \{1, \ldots, K\} \;\Rightarrow\; (x^K, x^0) \notin P(R).$$

Suzumura consistency is weaker than transitivity and stronger than acyclicity. Moreover, transitivity and Suzumura consistency are equivalent in the presence of completeness. Quasi-transitivity and Suzumura consistency are independent unless the relation under consideration is complete, in which case quasi-transitivity is implied by Suzumura consistency because of the latter's equivalence to transitivity.

As shown by Suzumura (1976), Suzumura consistency is necessary and sufficient for a relation R to possess an ordering extension; that is, there exists an ordering R' such that $R \subseteq R'$ and $P(R) \subseteq P(R')$. This observation is a significant strengthening of Szpilrajn's (1930) well-known theorem who shows that transitivity is sufficient for the existence of an ordering extension. In analogy to the transitive closure of a relation R (that is, the smallest transitive relation that contains R), Suzumura consistency allows for the existence of a well-defined closure operation; see Bossert, Sprumont, and Suzumura (2005). As is the case for the transitive closure, the Suzumura consistent closure of a relation R is the smallest Suzumura consistent relation that contains R. Suzumura (1978, 1999, 2000) applies Suzumura consistency the problem of right assignments and new welfare economics; see Bossert and Suzumura (2010) for a detailed discussion of Suzumura consistency and further applications.

3 Collective choice rules

The set of individuals is denoted by $N \subseteq \mathbb{N}$. The population N may be non-empty and finite or countably infinite. Each individual $i \in N$ is assumed to assess the alternatives in X by means of an ordering $R_i \subseteq X \times X$. A profile $\mathbf{R}$ is a list of orderings, one for each member of society. That is, $\mathbf{R} = (R_i)_{i \in N} \in \mathcal{R}^N$. Analogously to our notation for the restriction of a relation R to a pair of alternatives $\{x, y\}$, $\mathbf{R}|_{\{x,y\}}$ denotes the restriction of a profile $\mathbf{R}$ to $\{x, y\}$, that is, $\mathbf{R}|_{\{x,y\}} = (R_i|_{\{x,y\}})_{i \in N}$.

A collective choice rule $f : \mathcal{R}^N \to \mathcal{B}$ is a mapping that assigns a binary relation to each profile. We refer to f as a complete (an acyclical, a Suzumura consistent) collective choice rule if $f(\mathbf{R})$ is complete (acyclical, Suzumura consistent) for all $\mathbf{R} \in \mathcal{R}^N$.

We conclude this section with the definitions of the axioms that play a role in this paper.

Non-null. There exist $\mathbf{R} \in \mathcal{R}^N$ and $x, y \in X$ such that $(x, y) \in P(f(\mathbf{R}))$.

A strengthening of non-null is the well-known weak Pareto principle.

Weak Pareto. For all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$, if $(x, y) \in P(R_i)$ for all $i \in N$, then $(x, y) \in P(f(\mathbf{R}))$.

The following axiom is proposed by Blau and Deb (1977).

Independence, neutrality, and monotonicity with respect to P. For all $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$ and for all $x, y, x', y' \in X$, if

$$[(x, y) \in P(R_i) \Rightarrow (x', y') \in P(R'_i) \text{ and } (y', x') \in P(R'_i) \Rightarrow (y, x) \in P(R_i)] \text{ for all } i \in N,$$

then

$$(x, y) \in P(f(\mathbf{R})) \Rightarrow (x', y') \in P(f(\mathbf{R}')).$$

Note that the consequent of this axiom utilizes the asymmetric parts $P(f(\mathbf{R}))$ and $P(f(\mathbf{R}'))$ of the requisite social relations. Thus, it implies neutrality if a collective choice rule f is complete as assumed by Blau and Deb (1977). However, without the assumption of completeness of f, this axiom does not imply neutrality, as will be demonstrated later in the paper. The following (new) variant of the property employs the relations $f(\mathbf{R})$ and $f(\mathbf{R}')$ themselves instead. Therefore, it implies neutrality without the assumption of completeness of f. Since we analyze a Suzumura consistent collective choice rule f without assuming that f is complete, this axiom is a natural variant of the original property of Blau and Deb (1977).

Independence, neutrality, and monotonicity with respect to R. For all $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$ and for all $x, y, x', y' \in X$, if

$$[(x, y) \in P(R_i) \Rightarrow (x', y') \in P(R_i') \text{ and } (y', x') \in P(R_i') \Rightarrow (y, x) \in P(R_i)] \text{ for all } i \in N,$$

then

$$(x, y) \in f(\mathbf{R}) \Rightarrow (x', y') \in f(\mathbf{R}').$$

The above two axioms are independent, as will be demonstrated later in the paper. Specifically, the monotonicity property embodied by the latter only requires the monotonic preservation of a weak social relation and it does not guarantee the preservation of a strict social relation. It should be noted, however, that when we establish a possibility result, both versions of independence, neutrality, and monotonicity are employed in the list of properties.

Arrow (1951; 1963; 2012) imposes the property of non-dictatorship which requires that, for all $i \in N$, there exist $\mathbf{R} \in \mathcal{R}^N$ and $x, y \in X$ such that $(x, y) \in P(R_i)$ and $(x, y) \notin P(f(\mathbf{R}))$. The following axiom is a strengthening of Arrow's original condition.

Strong non-dictatorship. For all $i \in N$, there exist $\mathbf{R} \in \mathcal{R}^N$ and $x, y \in X$ such that $(x, y) \in P(R_i)$ and $(y, x) \in f(\mathbf{R})$.

Strong non-dictatorship implies non-dictatorship. The reverse implication is not valid unless social relations are required to be complete, in which case the two properties are equivalent.

Individual $i \in N$ is a vetoer if, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$, if $(x, y) \in P(R_i)$, then $(x, y) \in f(\mathbf{R})$. The following condition requires that there be no vetoer.

No veto. For all $i \in N$, there exist $\mathbf{R} \in \mathcal{R}^N$ and $x, y \in X$ such that $(x, y) \in P(R_i)$ and $(x, y) \notin f(\mathbf{R})$.

4 Finite populations

This section focuses on the case where the set of individuals is finite. In particular, we assume that $N = \{1, \ldots, n\}$ for some positive integer $n \in \mathbb{N}$.

An important result by Blau and Deb (1977) establishes that independence, neutrality, and monotonicity with respect to P implies the existence of a vetoer if a collective choice rule is complete and acyclical and the number of alternatives is greater than or equal to the number of individuals; see Sen (1986) for a concise proof. No additional properties such as weak Pareto are required.

Theorem 1. [Blau and Deb (1977)]. *Suppose that $|X| \geq |N|$. There exists no complete and acyclical collective choice rule f that satisfies independence, neutrality, and monotonicity with respect to P, and no veto.*

A natural question that emerges in this context is what happens if completeness is dropped and acyclicity is strengthened to Suzumura consistency. Because of the absence of completeness, some additional requirements are added in the following impossibility result. In particular, non-null and strong non-dictatorship are employed and, in addition, the alternative variant of independence, neutrality, and monotonicity that involves the social relation $f(\mathbf{R})$ rather than its asymmetric part $P(f(\mathbf{R}))$ is employed. We obtain the following result.

Theorem 2. *Suppose that $|X| \geq |N|$. There exists no Suzumura consistent collective choice rule f that satisfies non-null, independence, neutrality, and monotonicity with respect to R, and strong non-dictatorship.*

Proof. By way of contradiction, suppose that f is a Suzumura consistent collective choice rule that satisfies non-null, independence, neutrality, and monotonicity with respect to R, and strong non-dictatorship.

Since f is non-null, there exist $\mathbf{R}^* \in \mathcal{R}^N$ and $x, y \in X$ such that $(x, y) \in P(f(\mathbf{R}^*))$. Independence, neutrality, and monotonicity with respect to R implies that, for all $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$ and for all $x, y, x', y' \in X$, if

$$[(x, y) \in P(R_i) \Leftrightarrow (x', y') \in P(R'_i) \text{ and } (y', x') \in P(R'_i) \Leftrightarrow (y, x) \in P(R_i)] \text{ for all } i \in N,$$

then

$$(x, y) \in f(\mathbf{R}) \Leftrightarrow (x', y') \in f(\mathbf{R}').$$

Therefore, for all $x, y \in X$ such that $x \neq y$, there exists $\overline{\mathbf{R}} \in \mathcal{R}^N$ such that $(x, y) \in P(f(\overline{\mathbf{R}}))$. Setting $x' = x$ and $y' = y$ in the definition of independence, neutrality, and monotonicity with respect to R, it follows that, for all $\mathbf{R} \in \mathcal{R}^N$,

$$\mathbf{R}|_{\{x,y\}} = \overline{\mathbf{R}}|_{\{x,y\}} \Rightarrow (x, y) \in P(f(\mathbf{R})).$$

Strong non-dictatorship implies that, for each $i \in N$, there exist $\hat{\mathbf{R}} \in \mathcal{R}^N$ and $x, y \in X$ such that

$$(x, y) \in P(\hat{R}_i) \text{ and } (y, x) \in f(\hat{\mathbf{R}}).$$

Independence, neutrality, and monotonicity with respect to R implies that, for any $\mathbf{R} \in \mathcal{R}^N$ and any $w, z \in X$, if $(w, z) \in P(R_i)$ and $(z, w) \in P(R_j)$ for all $j \in N \setminus \{i\}$, then $(z, w) \in f(\mathbf{R})$.

Because $|X| \geq |N|$ by assumption, we can choose a profile of orderings $\mathbf{R}'' \in \mathcal{R}^N$ and alternatives $x^1, \ldots, x^{|N|} \in X$ such that

$$\begin{aligned}
&(x^1, x^2) \in P(R''_1), \ldots, (x^{|N|-1}, x^{|N|}) \in P(R''_1);\\
&(x^2, x^3) \in P(R''_2), \ldots, (x^{|N|-1}, x^{|N|}) \in P(R''_2), (x^{|N|}, x^1) \in P(R''_2);\\
&\quad\vdots\\
&(x^{|N|}, x^1) \in P(R''_{|N|}), (x^1, x^2) \in P(R''_{|N|}), \ldots, (x^{|N|-2}, x^{|N|-1}) \in P(R''_{|N|}).
\end{aligned}$$

By definition, for all $k \in \{2, \ldots, |N|\}$,

$$(x^k, x^{k-1}) \in P(R''_k) \text{ and } (x^{k-1}, x^k) \in P(R''_j) \text{ for all } j \in N \setminus \{k\}$$

and

$$(x^1, x^{|N|}) \in P(R_1'') \text{ and } (x^{|N|}, x^1) \in P(R_j'') \text{ for all } j \in N \setminus \{1\}.$$

It follows that

$$(x^{k-1}, x^k) \in f(\mathbf{R}'') \text{ for all } k \in \{2, \ldots, |N|\}$$

and

$$(x^{|N|}, x^1) \in f(\mathbf{R}'').$$

By Suzumura consistency, it follows that

$$(x^{k-1}, x^k) \in I(f(\mathbf{R}'')) \text{ for all } k \in \{2, \ldots, |N|\}$$

and

$$(x^{|N|}, x^1) \in I(f(\mathbf{R}'')).$$

Thus, independence, neutrality, and monotonicity with respect to R implies that, for all $\mathbf{R} \in \mathcal{R}^N$, for all $w, z \in X$, and for all $i \in N$,

$$(w, z) \in P(R_i) \text{ and } (z, w) \in P(R_j) \text{ for all } j \in N \setminus \{i\} \;\Rightarrow\; (z, w) \in I(f(\mathbf{R})). \tag{1}$$

Now it follows that there exists $\mathbf{R}^* \in \mathcal{R}^N$ such that $(x^1, x^2) \in P(f(\mathbf{R}^*))$ and, for all $\mathbf{R} \in \mathcal{R}^N$,

$$\mathbf{R}|_{\{x^1,x^2\}} = \mathbf{R}^*|_{\{x^1,x^2\}} \;\Rightarrow\; (x^1, x^2) \in P(f(\mathbf{R})). \tag{2}$$

Let $\mathbf{R}''' \in \mathcal{R}^N$ be such that

$$(x^1, x^3) \in P(R_1'''), (x^2, x^3) \in P(R_1'''), \ldots, (x^{|N|-1}, x^{|N|}) \in P(R_1''');$$

and, for all $i \in N \setminus \{1\}$,

$$(x^{|N|}, x^{|N|-1}) \in P(R_i'''), \ldots, (x^3, x^2) \in P(R_i'''), (x^3, x^1) \in P(R_i'''),$$

and

$$\mathbf{R}'''|_{\{x^1,x^2\}} = \mathbf{R}^*|_{\{x^1,x^2\}}.$$

By (2),

$$(x^1, x^2) \in P(f(\mathbf{R}''')). \tag{3}$$

Moreover, for all $k \in \{2, \ldots, |N| - 1\}$,

$$(x^k, x^{k+1}) \in P(R_1''') \text{ and } (x^{k+1}, x^k) \in P(R_j''') \text{ for all } j \in N \setminus \{1\}$$

and

$$(x^1, x^{|N|}) \in P(R_1''') \text{ and } (x^{|N|}, x^1) \in P(R_j''') \text{ for all } j \in N \setminus \{1\}.$$

From (1), it follows that

$$(x^2, x^3) \in I(f(\mathbf{R}''')), \ldots, (x^{|N|-1}, x^{|N|}) \in I(f(\mathbf{R}''')), (x^{|N|}, x^1) \in I(f(\mathbf{R}''')).$$

Together with (3), this contradicts Suzumura consistency. ■

As indicated earlier, there are several important differences between our Theorem 2 and Theorem 1, the result of Blau and Deb (1977). To recapitulate, we note first that Theorem 2 uses

Suzumura consistency in place of completeness and acyclicity. Moreover, our result imposes independence, neutrality, and monotonicity with respect to R as opposed to independence, neutrality, and monotonicity with respect to P. Unlike Blau and Deb (1977), we add the non-null property. Finally, the no veto axiom of Theorem 1 is replaced by strong non-dictatorship in Theorem 2.

An immediate corollary of Theorem 2 is that the impossibility persists if the stronger axiom of weak Pareto replaces non-null. Thus, we obtain

Corollary 1. *Suppose that $|X| \geq |N|$. There exists no Suzumura consistent collective choice rule f that satisfies weak Pareto, independence, neutrality, and monotonicity with respect to R, and strong non-dictatorship.*

To show that the axioms and assumptions of Theorem 2 and Corollary 1 are independent, we provide five collective choice rules, each of which satisfies all but one of them.

The Pareto rule f^P is defined by letting, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$(x, y) \in f^P(\mathbf{R}) \;\Leftrightarrow\; (x, y) \in \bigcap_{i \in N} R_i.$$

The Pareto extension rule f^{PE} is defined by letting, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$(x, y) \in f^{PE}(\mathbf{R}) \;\Leftrightarrow\; (y, x) \notin P(f^P(\mathbf{R}))$$

so that $P(f^{PE}(\mathbf{R})) = P(f^P(\mathbf{R}))$ and $I(f^{PE}(\mathbf{R})) = X \times X \setminus P(f^P(\mathbf{R}))$. The Pareto extension rule satisfies weak Pareto (and thus non-null), independence, neutrality, and monotonicity with respect to R, and strong non-dictatorship. The rule is not Suzumura consistent.

The null rule f^N is defined by letting, for all $\mathbf{R} \in \mathcal{R}^N$,

$$f^N(\mathbf{R}) = X \times X.$$

The null rule is not non-null (and, therefore, it does not satisfy weak Pareto). The rule is Suzumura consistent and satisfies independence, neutrality, and monotonicity with respect to R as well as strong non-dictatorship.

Fix two distinct alternatives $x^*, y^* \in X$, and define the collective choice rule f^V by letting, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$(x, y) \in f^V(\mathbf{R}) \Leftrightarrow \begin{cases} (y, x) \notin \bigcap_{i \in N} P(R_i) & \text{if } \{x, y\} = \{x^*, y^*\}, \\ (x, y) \in \bigcap_{i \in N} P(R_i) & \text{otherwise.} \end{cases}$$

It follows that, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$(x, y) \in P(f^V(\mathbf{R})) \;\Leftrightarrow\; (x, y) \in \bigcap_{i \in N} P(R_i)$$

and

$$(x, y) \in I(f^V(\mathbf{R})) \;\Leftrightarrow\; \left[(x, y) \notin \bigcap_{i \in N} P(R_i) \text{ and } (y, x) \notin \bigcap_{i \in N} P(R_i) \text{ and } \{x, y\} = \{x^*, y^*\} \right].$$

This rule satisfies weak Pareto, independence, neutrality, and monotonicity with respect to P, and strong non-dictatorship. To see that f^V is Suzumura consistent, observe that, for all $k \in \mathbb{N}\setminus\{1,2\}$, if $(x^\ell, x^{\ell+1}) \in f^V(\mathbf{R})$ for all $\ell \in \{1,\ldots,k-1\}$, then there exists $i \in N$ such that $(x^1, x^k) \in P(R_i)$. However, the rule violates independence, neutrality, and monotonicity with respect to R.

The Pareto rule is transitive and, thus, Suzumura consistent. Moreover, it satisfies weak Pareto, independence, neutrality, and monotonicity with respect to R, and no veto. The rule f^P does not satisfy strong non-dictatorship.

To show that the assumption that $|X| \geq |N|$ is necessary, suppose that $|X| < |N|$ and define the collective choice rule f^S by letting, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$(x,y) \in f^S(\mathbf{R}) \;\Leftrightarrow\; |\{i \in N \mid (x,y) \in P(R_i)\}| \geq |N| - 1.$$

This rule is a specific member of the class of S-rules that is axiomatized by Bossert and Suzumura (2008). Under the assumption $|X| < |N|$, it is a Suzumura consistent collective choice rule that satisfies weak Pareto, independence, neutrality, and monotonicity with respect to R, and strong non-dictatorship.

As noted earlier, the properties of independence, neutrality, and monotonicity with respect to P and independence, neutrality, and monotonicity with respect to R are independent. As shown above, the collective choice rule f^V satisfies independence, neutrality, and monotonicity with respect to P and violates independence, neutrality, and monotonicity with respect to R. To obtain a collective choice rule that satisfies independence, neutrality, and monotonicity with respect to R and violates independence, neutrality, and monotonicity with respect to P, fix $i^* \in N$ and define the collective choice rule f^* by letting, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$(x,y) \in f^*(\mathbf{R}) \;\Leftrightarrow\; \left[(x,y) \in \bigcap_{i \in N\setminus\{i^*\}} P(R_i) \text{ and } (x,y) \in I(R_{i^*})\right] \text{ or } (x,y) \notin I(R_{i^*}).$$

5 Countably infinite populations

This section examines the case in which the set of individuals is countably infinite; this allows us to assume, without loss of generality, that $N = \mathbb{N}$.

We begin with a possibility result, stating that the axioms of Theorem 2 are compatible if the population is countably infinite. To define the collective choice rule used in this result, we require the notion of a free ultrafilter.

An ultrafilter on N is a collection Ω of subsets of N such that

(u.i) $\varnothing \notin \Omega$;

(u.ii) For all $M \subseteq N, M \in \Omega$ or $N \setminus M \in \Omega$;

(u.iii) For all $M, M' \in \Omega, M \cap M' \in \Omega$.

An immediate consequence of the conjunction of (u.i) and (u.ii) is that $N \in \Omega$ for any ultrafilter Ω on N. Moreover, any ultrafilter Ω on N satisfies the following property.

$$\text{For all } M, M' \subseteq N, [M \in \Omega \text{ and } M \subseteq M' \Rightarrow M' \in \Omega]. \tag{4}$$

See, for example, Bossert and Suzumura (2010, Theorem 2.10).

Let Ω be an ultrafilter on N. If there exists $i \in N$ such that, for all $M \subseteq N$, $M \in \Omega$ if and only if $i \in M$, then Ω is a principal ultrafilter. Otherwise, Ω is a free ultrafilter. It is well-known that if N is finite, then all ultrafilters on N are principal. However, free ultrafilters do exist if N is infinite; see Willard (1970) for the construction of free ultrafilters.

Our possibility result is stated in the following theorem.

Theorem 3. *Suppose that $N = \mathbb{N}$. There exists a Suzumura consistent collective choice rule f that satisfies weak Pareto, independence, neutrality, and monotonicity with respect to P, independence, neutrality, and monotonicity with respect to R, and strong non-dictatorship.*

Proof. Let Ω be a free ultrafilter on $N = \mathbb{N}$, and define the collective choice rule f by

$$(x, y) \in f(\mathbf{R}) \;\Leftrightarrow\; \{i \in N \mid (x, y) \in R_i\} \in \Omega$$

for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$.

As shown by Hansson (1976, Theorem 2), $f(\mathbf{R})$ is transitive (hence Suzumura consistent) and complete for all $\mathbf{R} \in \mathcal{R}^N$. The same result shows that f satisfies weak Pareto and non-dictatorship. By completeness, strong non-dictatorship is equivalent to non-dictatorship so that this requirement is satisfied as well.

To prove that independence, neutrality, and monotonicity with respect to P is satisfied, let $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$ and $x, y, x', y' \in X$ be such that

$$[(x, y) \in P(R_i) \Rightarrow (x', y') \in P(R'_i) \;\text{ and }\; (y', x') \in P(R'_i) \Rightarrow (y, x) \in P(R_i)] \;\text{ for all } i \in N,$$

and $(x, y) \in P(f(\mathbf{R}))$. By definition, we have

$$\{i \in N \mid (x, y) \in R_i\} \in \Omega \;\text{ and }\; \{i \in N \mid (y, x) \in R_i\} \notin \Omega.$$

Because the individual relations are complete, it follows that

$$[(y', x') \in P(R'_i) \Rightarrow (y, x) \in P(R_i)] \Leftrightarrow [(x, y) \in R_i \Rightarrow (x', y') \in R'_i]$$

and hence $\{i \in N \mid (x, y) \in R_i\} \subseteq \{i \in N \mid (x', y') \in R'_i\}$. By (4),

$$\{i \in N \mid (x', y') \in R'_i\} \in \Omega$$

so that $(x', y') \in f(\mathbf{R}')$. Using the completeness of the individual relations again, it follows that

$$[(x, y) \in P(R_i) \Rightarrow (x', y') \in P(R'_i)] \;\Leftrightarrow\; [(y', x') \in R'_i \Rightarrow (y, x) \in R_i].$$

Thus,

$$\{i \in N \mid (y', x') \in R'_i\} \subseteq \{i \in N \mid (y, x) \in R_i\}.$$

If $(y', x') \in f(\mathbf{R}')$, $\{i \in N \mid (y', x') \in R'_i\} \in \Omega$, and (4) implies $\{i \in N \mid (y, x) \in R_i\} \in \Omega$ which, in turn, implies $(y, x) \in f(\mathbf{R})$. This contradicts the assumption that $(x, y) \in P(f(\mathbf{R}))$ and, therefore, it must be the case that $(y', x') \notin f(\mathbf{R}')$. It follows that $(x', y') \in P(f(\mathbf{R}'))$ so that independence, neutrality, and monotonicity with respect to P is satisfied.

Finally, we show that f satisfies independence, neutrality, and monotonicity with respect to R. Let $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$ and $x, y, x', y' \in X$ be such that

$$[(x, y) \in P(R_i) \Rightarrow (x', y') \in P(R'_i) \;\text{ and }\; (y', x') \in P(R'_i) \Rightarrow (y, x) \in P(R_i)] \;\text{ for all } i \in N,$$

and $(x, y) \in f(\mathbf{R})$. By definition, we have

$$\{i \in N \mid (x, y) \in R_i\} \in \Omega.$$

Because the individual relations are complete,

$$[(y', x') \in P(R'_i) \Rightarrow (y, x) \in P(R_i)] \Leftrightarrow [(x, y) \in R_i \Rightarrow (x', y') \in R'_i].$$

Now it follows that $\{i \in N \mid (x, y) \in R_i\} \subseteq \{i \in N \mid (x', y') \in R'_i\}$. By (4), we obtain

$$\{i \in N \mid (x', y') \in R'_i\} \in \Omega.$$

This implies $(x', y') \in f(\mathbf{R}')$ and, therefore, independence, neutrality, and monotonicity with respect to R is satisfied. ■

Theorem 3 uses the properties of Theorem 2 to illustrate that the impossibility disappears in the countably infinite case. It is immediate that a stronger possibility result is valid because the social relation defined in the proof of Theorem 3 is transitive and complete rather than merely Suzumura consistent.

If X is countably infinite, strong non-dictatorship cannot be extended to a coalitional variant in Theorem 3. In this case, the stronger axiom leads to an impossibility result even if independence, neutrality, and monotonicity with respect to P is removed.

To formulate the axiom of strong coalitional non-dictatorship in the countably infinite setting, we first define, for all $A \subseteq \mathbb{N}$ and for all $k \in \mathbb{N}$,

$$\alpha'(A; k) = \frac{|A \cap \{1, \ldots, k\}|}{k}.$$

The asymptotic density of A is given by

$$\alpha(A) = \lim_{k \to \infty} \alpha'(A; k),$$

provided that this limit exists. We note that there are sets $A \subseteq N$ for which the limit does not exist; in these cases the asymptotic density is not defined.

Strong coalitional non-dictatorship. There exists $\varepsilon \in \mathbb{R}_{++}$ such that, for all $A \subseteq N$ with $\alpha(A) < \varepsilon$, there exist $\mathbf{R} \in \mathcal{R}^N$ and $x, y \in X$ such that $(x, y) \in P(R_i)$ for all $i \in A$ and $(y, x) \in f(\mathbf{R})$.

See Cato (2017) for the axiom of coalitional non-dictatorship which also employs asymptotic densities.

We conclude this section with the following impossibility theorem.

Theorem 4. *Suppose that $N = \mathbb{N}$ and $|X| = |\mathbb{N}|$. There exists no Suzumura consistent collective choice rule f that satisfies weak Pareto, independence, neutrality, and monotonicity with respect to R, and strong coalitional non-dictatorship.*

Proof. Let f be a Suzumura consistent collective choice rule that satisfies weak Pareto, independence, neutrality, and monotonicity with respect to R, and strong coalitional non-dictatorship.

Strong coalitional non-dictatorship implies that there exists $\varepsilon > 0$ such that, for all $A \subseteq N$ with $\alpha(A) < \varepsilon$, there exist $\mathbf{R} \in \mathcal{R}^N$ and $x, y \in X$ such that $(x, y) \in P(R_i)$ for all $i \in A$ and $(y, x) \in f(\mathbf{R})$.

Let $K \in \mathbb{N}$ be such that $1/K < \varepsilon$ and define

$$A^k = \{i \in \mathbb{N} \mid \exists n \in \mathbb{N} \text{ such that } i = k + K(n-1)\}$$

for all $k \in \{1, \ldots, K\}$. More explicitly, these sets are given by

$$\begin{aligned} A^1 &= \{1, K+1, 2K+1, 3K+1, \ldots\}, \\ A^2 &= \{2, K+2, 2K+2, 3K+2, \ldots\}, \\ &\vdots \\ A^K &= \{K, 2K, 3K, 4K, \ldots\}. \end{aligned}$$

The asymptotic densities are $\alpha(A^k) = 1/K$ so that $\alpha(A^k) < \varepsilon$ for all $k \in \{1, \ldots, K\}$. Thus, for each $k \in \{1, \ldots, K\}$, there exist $\mathbf{R} \in \mathcal{R}^N$ and $x, y \in X$ such that $(x, y) \in P(R_i)$ for all $i \in A^k$ and $(y, x) \in f(\mathbf{R})$.

The axiom of independence, neutrality, and monotonicity with respect to R implies that, for all $k \in \{1, \ldots, K\}$, for all $\mathbf{R}' \in \mathcal{R}^N$, and for all $w, z \in X$, if $(w, z) \in P(R'_i)$ for all $i \in A^k$ and $(z, w) \in P(R'_j)$ for all $j \in N \setminus A^k$, then $(z, w) \in f(\mathbf{R}')$.

Let $x^0, \ldots, x^K \in X$. Because f satisfies weak Pareto and hence non-null, the same argument that is employed in the proof of Theorem 2 can be used to conclude that there exists $\mathbf{R}^* \in \mathcal{R}^N$ such that $(x^K, x^0) \in P(f(\mathbf{R}^*))$ and, for all $\mathbf{R} \in \mathcal{R}^N$,

$$\mathbf{R}|_{\{x^0, x^K\}} = \mathbf{R}^*|_{\{x^0, x^K\}} \;\Rightarrow\; (x^K, x^0) \in P(f(\mathbf{R})).$$

Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\begin{aligned} &(x^1, x^2) \in P(R_i), \ldots, (x^{K-1}, x^K) \in P(R_i), (x^K, x^0) \in P(R_i) \text{ for all } i \in A^1; \\ &(x^2, x^3) \in P(R_i), \ldots, (x^{K-1}, x^K) \in P(R_i), (x^K, x^0) \in P(R_i), (x^0, x^1) \in P(R_i) \text{ for all } i \in A^2; \\ &\quad\vdots \\ &(x^K, x^0) \in P(R_i), (x^0, x^1) \in P(R_i), \ldots, (x^{K-2}, x^{K-1}) \in P(R_i) \text{ for all } i \in A^K \end{aligned}$$

and

$$\mathbf{R}|_{\{x^0, x^K\}} = \mathbf{R}^*|_{\{x^0, x^K\}}.$$

By definition, for all $k \in \{1, \ldots, K\}$,

$$(x^k, x^{k-1}) \in P(R_i) \text{ for all } i \in A^k \text{ and } (x^{k-1}, x^k) \in P(R_j) \text{ for all } j \in N \setminus A^k.$$

We obtain

$$(x^{k-1}, x^k) \in f(\mathbf{R}) \text{ for all } k \in \{1, \ldots, K\}$$

and, because $\mathbf{R}|_{\{x^0, x^K\}} = \mathbf{R}^*|_{\{x^0, x^K\}}$, weak Pareto implies that

$$(x^K, x^0) \in P(f(\mathbf{R})).$$

This contradicts Suzumura consistency. ■

The axioms and assumptions of Theorem 4 are independent. For any set $A \subseteq N$, we write the complement of A in N as $A^c = N \setminus A$.

The infinite-population extensions of the Pareto extension rule f^{PE}, the null rule f^N, the collective choice rule f^V, and the Pareto rule f^P, respectively, can be used to show that the assumption of Suzumura consistency and each of the axioms of weak Pareto, independence, neutrality, and monotonicity with respect to R, and strong coalitional non-dictatorship is not implied by the remaining properties.

To show that the assumption $|X| = |\mathbb{N}|$ is necessary for establishing Theorem 4, suppose that $N = \mathbb{N}$ and $|X| < \infty$. We define the collective choice rule $f^{S'}$ by letting, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$(x, y) \in f^{S'}(\mathbf{R}) \;\Leftrightarrow\; \exists A \subseteq \{i \in N \mid (x, y) \in P(R_i)\} \text{ such that } \alpha(A) > 1 - \frac{1}{|X|}.$$

This rule is an infinite-population variant of an S-rule of Bossert and Suzumura (2008). It satisfies weak Pareto and independence, neutrality, and monotonicity with respect to R. To show that it satisfies strong coalitional non-dictatorship, let $\varepsilon = 1/|X|$ and $A \subseteq N$ with $0 < \alpha(A) < \varepsilon$. We obtain

$$\alpha(A^c) = \lim_{k\to\infty} \frac{k - |A \cap \{1, \ldots, k\}|}{k} = 1 - \alpha(A) > 1 - \frac{1}{|X|}.$$

For any $x, y \in X$, there exists $\mathbf{R} \in \mathcal{R}^N$ such that $(x, y) \in P(R_i)$ for all $i \in A$ and $(y, x) \in P(R_j)$ for all $j \in A^c$. By definition, $(y, x) \in f^{S'}(\mathbf{R})$. Thus, $f^{S'}$ satisfies strong coalitional non-dictatorship. Furthermore, it is Suzumura consistent. This can be verified as follows. Let $\mathbf{R} \in \mathcal{R}^N$, $K \in \mathbb{N} \setminus \{1, 2\}$ with $K \leq |X|$, and $x^1, \ldots, x^K \in X$. Suppose that $(x^k, x^{k+1}) \in f^{S'}(\mathbf{R})$ for all $k \in 1, \ldots, K-1$. For all $k \in \{1, \ldots, K-1\}$, define A^k by

$$A^k = \{i \in N \mid (x^k, x^{k+1}) \in P(R_i)\}.$$

Moreover, define A by

$$A = \{i \in N \mid (x^K, x^1) \in P(R_i)\}.$$

We show that there exists no $A^* \subseteq A$ such that $\alpha(A^*) > 1 - 1/|X|$. By way of contradiction, suppose that such a set A^* exists. Note that, for each $k \in \{1, \ldots, K-1\}$, there exists $A^{k*} \subseteq A^k$ such that

$$\alpha(A^{k*}) > 1 - \frac{1}{|X|}.$$

and, thus,

$$\alpha((A^{k*})^c) < \frac{1}{|X|}.$$

Furthermore, it follows that

$$A^* \subseteq A \subseteq \bigcup_{k=1}^{K-1} (A^k)^c \subseteq \bigcup_{k=1}^{K-1} (A^{k*})^c.$$

Thus, we obtain

$$1 - \frac{1}{|X|} \geq \frac{K-1}{|X|} > \sum_{k=1}^{K-1} \alpha((A^{k*})^c) \geq \alpha\left(\bigcup_{k=1}^{K-1} (A^{k*})^c\right) \geq \alpha(A^*) > 1 - \frac{1}{|X|},$$

a contradiction.

6 Concluding remarks

Although somewhat overlooked initially after its introduction by Suzumura (1976), Suzumura consistency has proven to be a very useful property in the analysis of individual and collective choice, as demonstrated by contributions such as Cato (2013), Bossert and Suzumura (2015), and Bossert and Cato (2021), among others. It can actually be argued that, in the absence of completeness, Suzumura consistency is more natural than transitivity itself. It seems eminently reasonable to exclude the negative consequences and contradictory recommendations that emerge if cycles with at least one instance of betterness appear in a goodness relation. However, if non-comparabilities are a possibility to begin with, there seems to be no compelling reason to force an at-least-as-good-as relationship between two alternatives x and z on the basis of there being an alternative y that is at most as good as x and at least a good as z. If x is at least as good as y and y, in turn, is at least as good as z, it is perfectly acceptable that x and z are in a state of non-comparability. All that has to be avoided is that z be declared better than x—and this is what Suzumura consistency does.

The present paper serves to further illustrate that Suzumura consistency can be employed in combination with fundamental requirements on choice procedures—namely, the two variants of independence, neutrality, and monotonicity. In addition to the results themselves, we hope that the paper will turn out to be useful in that the new proof techniques it provides may find applications in other branches of the literature as well.

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