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Thresholds, critical levels, and generalized sufficientarian principles*

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Abstract. This paper provides an axiomatic analysis of sufficientarian social evaluation. Sufficientarianism has emerged as an increasingly important notion of distributive justice. We propose a class of principles that we label generalized critical-level sufficientarian orderings. These orderings provide a welfarist foundation for social policies whose objective is the reduction of poverty. The distinguishing feature of our new class is that its members exhibit constant critical levels of well-being that are allowed to differ from the threshold of sufficiency. Our basic axiom assigns absolute priority to those below the threshold, a property that is shared by numerous other sufficientarian approaches. When combined with the well-known strong Pareto principle and the assumption that there be a constant critical level, the axiom implies that the critical level cannot be below the threshold. The main results of the paper are characterizations of our new class and an important subclass. As a final observation, we identify the generalized critical-level sufficientarian orderings that permit us to avoid the repugnant conclusion and the sadistic conclusion, which are known as two fundamental challenges in population ethics. *Journal of Economic Literature* Classification Nos.: D31, D63.

Keywords: poverty, sufficientarianism, critical level, threshold, strong Pareto, population ethics.

1 Introduction

The incidence of poverty remains alarmingly high in many countries. Indeed, 10% of the world population live in a state of extreme poverty, and the fundamental needs of numerous people in developing countries fail to be satisfied. The two foremost objectives listed in the Sustainable Development Goals (UN General Assembly, 2015) are "no poverty" and "zero hunger;" see also Sachs (2012). Moreover, inequality is on the rise also in developed countries, including the United States and numerous European-Union member states. There has been a dramatic increase in the mortality rate of the middle class in the United States, and income levels are polarized because of technological changes and educational disparities; see Case and Deaton (2015) and Goldin and Katz (2008). As a result, there are now more and more people who are deemed not to have enough to achieve a minimally acceptable standard of living, both in developed and in developing countries.

One of the common ideas to overcome these severe insufficiencies is to use social safety nets to guarantee that everyone has a sufficiently high level of well-being. The need for a sound normative foundation of social-security policies has been an important issue in welfare economics for centuries; in fact, the issue goes back at least as far as Bentham (1789) and Pigou (1912). More recent approaches include those of Rawls (1971), Nozick (1974), and Meade (1976), to name but a few. A plausible objective of public policy is the choice of a social-security system that improves the quality of life of those whose well-being is deemed to be at a level below sufficiency and that guarantees as many people as possible to have enough to achieve a minimally acceptable standard of living. This objective cannot be fulfilled by restricting attention to poverty reduction. This is the case because poverty is typically evaluated with an exclusive focus on shortfalls from the poverty (or insufficiency) line and, therefore, poverty measures are unsuitable to provide unambiguous welfare assessments of entire distributions of well-being. This problem cannot be resolved by employing standard inequality-averse social welfare measures because these do not explicitly involve a threshold of sufficiency and, as a consequence, they cannot give priority to people below the threshold of sufficiency. Therefore, to address the issue of insufficiencies, we need an alternative approach to social evaluation which we may call insufficiency averse, and this is where the notion of sufficientarianism enters the picture.

This paper provides an axiomatic foundation of a class of social-evaluation orderings that are based on sufficientarian principles. Sufficientarianism is a theory of justice that has attracted considerable attention in political philosophy in the recent past. Its reach extends well beyond that field, however; in particular, it is highly relevant for numerous issues in welfare economics. The notion of sufficiency is based on the view that individuals should be at or above a given threshold of utility, which we interpret as an indicator of lifetime well-being. The seminal contribution in this area is that of Frankfurt (1987) who advocates the maximization of the number of individuals whose utilities are above the threshold. Frankfurt's proposal is closely related to the head-count ratio, a well-know measure of poverty. However, Frankfurt's sufficientarian account exhibits several fundamental shortcomings. The first difficulty is that his proposed ranking of distributions of utility is not consistent with the Pareto principle. Moreover, his criterion is not distributionally sensitive to an adequate degree; this latter observation does not come as surprise because it shares this property with the head-count ratio that it resembles. Indeed, Frankfurt's ranking may recommend a very unequal distribution when the resources available are insufficient for allowing everyone to be above the threshold. Moreover, Frankfurt's account of sufficiency is in conflict with the well-established Pigou-Dalton transfer principle (see Pigou, 1912, and Dalton, 1920).

Crisp (2003) recommends an interesting refinement of Frankfurt's (1987) proposal. In addition to giving absolute priority to those below the threshold, his principle pays due attention to distributional concerns below the threshold but assumes a neutral position towards redistributions that take place above the threshold; see Crisp (2003, p. 758) for details. Crisp's sufficientarian account is extended by Brown (2005), Huseby (2010, 2012), and Hirose (2016). As Crisp's proposal makes clear, sufficientarianism is related to but distinct from poverty measurement. A sufficientarian approach to social evaluation focuses primarily on those whose well-being is below the threshold level that represents sufficiency but it can be supplemented by additional criteria so that it is consistent with the Pareto principle.

The axiomatic analysis carried out in this paper which, in part, formalizes Crisp's (2003) proposal, is of relevance for numerous social and economic policy choices that may affect not only the current population but also future generations and their wellbeing. To assess population consequences, an essential criterion is that of a critical level of lifetime well-being, which is familiar from the literature on population ethics. If a person at this level is added to a utility-unaffected population, the expanded society is as good as the original. Phrased in the context of sufficientarian social evaluation, a crucial question arises immediately: what is an ethically appealing relationship between the sufficiency threshold and this critical level? Although all of the existing sufficientarian theories assume (at least implicitly) that the two coincide, there is no a priori reason why such fundamentally different values should be the same. Indeed, allowing them to be distinct gives us considerably more flexibility when it comes to the choice of policies that may affect population size.

We propose a general class of sufficientarian orderings which we label generalized critical-level sufficientarian orderings. These criteria satisfy absolute priority and are compatible with all of the properties advocated by Crisp (2003). In addition, they approximate the Frankfurt criterion as a limiting case. Two numbers play crucial roles in these orderings. The first is an externally given threshold of sufficiency, the second is a critical level of utility. A special case is obtained if the two values coincide, and we discuss the resulting critical-level sufficientarian orderings in our companion contribution Bossert, Cato, and Kamaga (2020). Critical-level sufficientarianism is inspired by Blackorby and Donaldson's (1984) critical-level generalized-utilitarian population principles; see Blackorby, Bossert, and Donaldson (2005) for a comprehensive analysis of critical-level generalized utilitarianism.

As a first result, we show that the critical level must be greater than or equal to the threshold of sufficiency. This is the case because if a constant critical level exists and is located below the threshold, strong Pareto is violated. Since the threshold for sufficiency is assumed to be associated with the fundamental needs of a human being (Braybrooke, 1987, and Wiggins, 1998), it is very natural to consider a critical utility level that is higher than the threshold.

Our main contribution is a characterization of the new class of generalized critical-level sufficientarian orderings. An additional result narrows down the class to a subclass that no longer contains Frankfurt's (1987) proposal as a limiting case. This is accomplished by adding a transfer principle across the threshold. The resulting subclass is more compact in that there is one less parameter to choose and its members have the attractive feature of respecting a transfer principle that is on as sound an ethical footing as the variants that are restricted to transfers below or above the threshold.

A notable feature of our orderings is that some of them can avoid two well-known undesirable attributes that have been studied in population ethics by authors such as Parfit (1976, 1982, 1984), Blackorby and Donaldson (1984), Arrhenius (2000, forthcoming), Blackorby, Bossert, and Donaldson (2004, 2005), and Spears and Budolfson (2019). If the critical level is higher than the utility level that represents a neutral life, Parfit's repugnant conclusion can be avoided. Moreover, if the threshold for sufficiency is equal to neutrality, then Arrhenius's sadistic conclusion does not materialize. It is worth emphasizing that both of these unattractive conclusions can be avoided in our setting. This is in stark contrast to the standard formulation of critical-level generalized utilitarianism in population ethics—it does not leave any room to avoid both.

The dilemma between avoiding the repugnant and sadistic conclusions has been examined by Blackorby, Bossert, and Donaldson (2004, 2005), Asheim and Zuber (2014), Zuber (2018), and Pivato (2020). Some of these contributions employ orderings that are based on utilitarian principles, while others focus on egalitarian orderings that give unequivocal priority to the worst-off, and those egalitarian orderings are shown to avoid both conclusions. In view of this contrast between utilitarian and egalitarian approaches, the reconciliation of the population-ethics dilemma by our orderings is worth emphasizing because they do not give exclusive priority to the worst-off. Instead, they focus on those with insufficient levels of well-being, thereby accommodating welfare gains and losses involving more than one person. Our orderings avoid the sadistic conclusion only if the threshold is equal to the neutral level. This implies that, unlike some of its generalizations, a critical-level sufficientarian ordering cannot avoid both conclusions because it requires that the critical level be equal to the threshold.

Section 2 introduces the setting and basic definitions. Our axioms are defined in Section 3, and Section 4 presents our results and their proofs. Section 5 examines some population-ethics properties of our orderings and compares them with other well-known classes of orderings. Section 6 concludes. The appendix establishes the independence of the axioms used in our main result.

2 Setting

A utility distribution for $n \in \mathbb{N}$ individuals is given by an n-dimensional vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$. Each u_i represents the level of lifetime well-being of individual i. The set of all possible distributions is given by $\Omega = \bigcup_{n \in \mathbb{N}} \mathbb{R}^n$.

A neutral life is a life that, from the viewpoint of the person leading it, is as good as a life without any experiences. We follow the standard convention in population ethics and normalize the level of utility that corresponds to neutrality to zero; see Blackorby and Donaldson (1984, p. 14) or Blackorby, Bossert, and Donaldson (2005, p. 25), for example. Thus, a life is worth living if lifetime utility is positive—above neutrality.

We employ a notion of sufficientarianism that can be captured by means of an ordering (that is, a reflexive, complete, and transitive binary relation) R defined on the set of distributions Ω. For notational convenience, we write uRv instead of $(u, v) \in R$ with the interpretation that distribution u is at least as good as distribution v from a sufficientarian perspective. The asymmetric and symmetric parts corresponding to R are denoted by P and I.

The fundamental ingredient of a sufficientarian analysis is an exogenously given threshold level of utility, denoted by $\theta \in \mathbb{R}$. Its interpretation is that individuals who experience at least this level of well-being have enough, whereas those whose utility is below θ do not. For all $n \in \mathbb{N}$ and for all $u \in \mathbb{R}^n$, we define

$$L^n(u) = \{i \in \{1, \ldots, n\} \mid u_i < \theta\};$$
$$E^n(u) = \{i \in \{1, \ldots, n\} \mid u_i = \theta\};$$
$$H^n(u) = \{i \in \{1, \ldots, n\} \mid u_i > \theta\}.$$

These are the sets of those whose lifetime utility is lower than, equal to, and higher than the threshold level θ. Let $\mathbb{R}^n_L = (-\infty, \theta)^n$, $\mathbb{R}^n_{LE} = (-\infty, \theta]^n$, and $\mathbb{R}^n_{HE} = [\theta, \infty)^n$, and define $\Omega_L = \cup_{n \in \mathbb{N}} \mathbb{R}^n_L$, $\Omega_{LE} = \cup_{n \in \mathbb{N}} \mathbb{R}^n_{LE}$, and $\Omega_{HE} = \cup_{n \in \mathbb{N}} \mathbb{R}^n_{HE}$. For notational simplicity, we write $\mathbb{R}_L$, $\mathbb{R}_{LE}$, and $\mathbb{R}_{HE}$ instead of $\mathbb{R}^1_L$, $\mathbb{R}^1_{LE}$, and $\mathbb{R}^1_{HE}$.

Taking the threshold θ of sufficiency of well-being as exogenously given, we refer to an ordering R on Ω as a sufficientarian ordering if it is intended to evaluate utility distributions from a sufficientarian perspective. In particular, this means that a sufficientarian ordering compares utility distributions with priority being given to those below the threshold. Our notion of a sufficientarian ordering is analogous to that of a poverty ordering with an exogenously given poverty line.

Let G be the set of all functions $g\colon \mathbb{R} \to \mathbb{R}$ that are increasing and strictly concave on $\mathbb{R}_L$, and continuous, increasing, and concave on $\mathbb{R}_{HE}$. As is well-known, because $\mathbb{R}_L$ is an open interval, the (strict) concavity of g on $\mathbb{R}_L$ implies that g is continuous on $\mathbb{R}_L$. An analogous observation does not apply to the half-open interval $\mathbb{R}_{HE}$—it is possible that an increasing and concave function is not continuous at the boundary point θ, which is why the requisite continuity property does not follow. An ordering R on Ω is generalized critical-level sufficientarian if there exist $g \in G$, $\delta_L \in \mathbb{R}$ with $\delta_L \geq \sup_{a \in \mathbb{R}_L} g(a)$, and $\delta_H \in \mathbb{R}$ with $g(\theta) \leq \delta_H < \sup_{a \in \mathbb{R}_{HE}} g(a)$ such that, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$, uRv if and only if

$$\sum_{i \in L^n(u)} [g(u_i) - \delta_L] > \sum_{i \in L^m(v)} [g(v_i) - \delta_L]$$

or

$$\sum_{i \in L^n(u)} [g(u_i) - \delta_L] = \sum_{i \in L^m(v)} [g(v_i) - \delta_L]$$

and

$$\sum_{i \in E^n(u) \cup H^n(u)} [g(u_i) - \delta_H] \geq \sum_{i \in E^m(v) \cup H^m(v)} [g(v_i) - \delta_H] .$$

Generalized critical-level sufficientarianism evaluates the value of a life as the shortfall of the transformed lifetime utility $g(u_i)$ from δ_L for a person i below the threshold, and as the difference between the transformed lifetime utility $g(u_j)$ and δ_H for a person at or above the threshold, where δ_L and δ_H may differ. The principle ranks utility distributions by applying a lexical criterion to the resulting sums of values. First, the sums of the values of people below the threshold are compared. If this results in a strict ranking, this ranking is adopted as the sufficientarian ranking of the two distributions in question. If the requisite sums are equal, the tie is broken by comparing the sums that correspond to those at or above the threshold. Note that the value of a life of a person at or above the threshold, $g(u_j) - \delta_H$, is not necessarily positive.

Without loss of generality, g can be chosen so that $g(0) = 0$—that is, the value of g at a neutral life is equal to zero; see, for example, Blackorby, Bossert, and Donaldson (2005, Theorem 4.7). We note that g is allowed to be discontinuous at θ because the limit of g as we approach θ from below does not have to be equal to the value of g at θ. For that reason, the function g need not be increasing on its entire domain $\mathbb{R}$—it is possible that $g(a) > g(\theta)$ for some values of a below the threshold.

By construction, we can find some utility value $\alpha \in [\theta, \infty)$ such that $g(\alpha) = \delta_H$. This value α is a critical level for this ordering and, thus, δ_H is the image of the critical level α under g. That is, it holds that, for all $u \in \Omega$, $uI(u, \alpha)$. Therefore, the tie-breaking step of evaluation by a generalized critical-level sufficientarian ordering consists of the application of the critical-level generalized-utilitarian ordering associated with a critical level α to the lifetime utilities of those at or above the threshold. If $g(\alpha) > g(\theta)$, the critical level is strictly higher than the threshold for sufficiency because g is increasing on $[\theta, \infty)$.

The role of δ_L is similar to that of δ_H. However, δ_L does not correspond to a critical level. Assume that there exists $\beta \in [\theta, \infty)$ such that $g(\beta) = \delta_L$. Note that such a value β need not exist and, even if it does, it cannot be associated with a critical level. This is the case because $uP(u, \beta)$ is true for all $u \in \Omega$ if $\alpha > \beta$, while $(u, \beta)Pu$ results for all $u \in \Omega$ if $\alpha < \beta$. For the special case in which $g(\theta) = \delta_H = \delta_L$ and g is continuous on its entire domain, R reduces to a critical-level sufficientarian ordering—the case examined in detail by Bossert, Cato, and Kamaga (2020).

3 Axioms

Our first axiom is a core principle of sufficientarianism, which states that if (a) the distribution that corresponds to the subgroup of those who are below the threshold is better for one distribution than another, then the former distribution is better than the latter overall; and (b) if there is equal goodness below the threshold, then the relative ranking of the two distributions overall is determined by the relative aggregate values assigned to those at and above the threshold.

Absolute priority. For all $n, m, \ell, r \in \mathbb{N}$, for all $u \in \mathbb{R}^n_L$, for all $v \in \mathbb{R}^m_L$, for all $w \in \mathbb{R}^\ell_{HE}$, and for all $s \in \mathbb{R}^r_{HE}$,

(a) $uPv \;\Rightarrow\; (u, w)P(v, s)$;

(b) if uIv, then $[(u, w)R(v, s) \;\Leftrightarrow\; wRs]$.

There is a subtle difference between this axiom and the variant that we employ in Bossert, Cato, and Kamaga (2020). In our companion contribution, we extend the range of possible values of u and v to $\mathbb{R}^n_{LE}$ and $\mathbb{R}^m_{LE}$ rather that restricting attention to $\mathbb{R}^n_L$ and $\mathbb{R}^m_L$, respectively. This difference is associated with the definition of the disadvantaged; in particular, it needs to be determined whether someone at the threshold is considered to be among those who are in need of special attention. This is parallel to the question of whether the weak or the strong definition of the poor is to be employed in the measurement of poverty. As shown by Donaldson and Weymark (1986), if people located on the poverty line are treated as poor, it is impossible to construct a poverty measure that satisfies some normatively desirable axioms. From this perspective, the weak definition of the poor is a more acceptable definition, and our definition of the disadvantaged conforms to this observation.

The following anonymity requirement states that, for any distribution, all of its permuted distributions are as good as the original. Thus, the axiom represents a fundamental equal-treatment property in a welfarist framework.

Anonymity. For all $n \in \mathbb{N}$ and for all $u, v \in \mathbb{R}^n$, if v is obtained by applying a permutation to the components of u, then vIu.

Another axiom that needs little explanation is the strong Pareto principle. According to this property, a change is normatively desirable if it makes all individuals weakly better off and at least one individual strictly better off.

Strong Pareto. For all $n \in \mathbb{N}$ and for all $u, v \in \mathbb{R}^n$, if $u_i \geq v_i$ for all $i \in \{1, \ldots, n\}$ with at least one strict inequality, then uPv.

The following condition is a strong independence axiom familiar from the literature on population ethics; see, for example, Blackorby, Bossert, and Donaldson (2005, p. 159).

Existence independence. For all $u, v, w \in \Omega$,

$$uRv \;\Leftrightarrow\; (u, w)R(v, w).$$

We note that our orderings do not necessarily satisfy continuity on the entire domain. If we consider the subdomain of the utilities of those below the threshold, the continuity requirement can be applied.

Continuity below the threshold. For all $n \in \mathbb{N}$ and for all $u \in \mathbb{R}^n_L$, the sets

$$\{v \in \mathbb{R}^n_L \mid vPu\} \text{ and } \{v \in \mathbb{R}^n_L \mid uPv\}$$

are open in $\mathbb{R}^n_L$.

The following counterpart of this property requires that the continuity requirement applies to the subdomain of the utilities of those who have enough.

Continuity above and at the threshold. For all $n \in \mathbb{N}$ and for all $u \in \mathbb{R}^n_{HE}$, the sets

$$\{v \in \mathbb{R}^n_{HE} \mid vRu\} \text{ and } \{v \in \mathbb{R}^n_{HE} \mid uRv\}$$

are closed in $\mathbb{R}^n_{HE}$.

Some additional properties are required for our purposes. The first of these is very weak—it merely requires that a critical level exists for at least one distribution. It appears in Blackorby, Bossert, and Donaldson (2005, p. 160).

Weak existence of critical levels. There exist $u \in \Omega$ and $\alpha \in \mathbb{R}$ such that

$$uI(u, \alpha).$$

The next axiom is a strengthening of this property, which first appeared in Blackorby and Donaldson (1984). It requires the existence of a utility value that can be applied to any distribution as a critical level.

Existence of constant critical levels. There exists $\alpha \in \mathbb{R}$ such that, for all $u \in \Omega$,

$$uI(u, \alpha).$$

Our next axiom is new; it is one of the core axioms of our characterization of the class of generalized critical-level sufficientarian orderings. Consider any two population sizes n and m and two distributions $u \in \mathbb{R}^n_L$ and $v \in \mathbb{R}^m_L$. The condition requires the existence of two utility levels λ and μ below the threshold such that if multiple individuals with utility λ are added to any $u \in \mathbb{R}^n_L$ and multiple individuals with utility μ are added to any $v \in \mathbb{R}^m_L$, the relative ranking of the augmented distributions is the same as the relative ranking of u and v. The multiples are chosen so that the population sizes in the two augmented distributions are the same.

Expansion independence. For all $n, m \in \mathbb{N}$, there exist $k \in \mathbb{N}$ with $k > \max\{n, m\}$ and $\lambda, \mu \in \mathbb{R}_L$ such that, for all $u \in \mathbb{R}^n_L$ and for all $v \in \mathbb{R}^m_L$,

$$uRv \;\Leftrightarrow\; (u, \underbrace{\lambda, \ldots, \lambda}_{k-n})R(v, \underbrace{\mu, \ldots, \mu}_{k-m})$$

and

$$vRu \;\Leftrightarrow\; (v, \underbrace{\mu, \ldots, \mu}_{k-m})R(u, \underbrace{\lambda, \ldots, \lambda}_{k-n}).$$

Strictly speaking, the values k, λ, and μ may depend on n and m and, therefore, the use of the symbols $k_{n,m}$, $\lambda_{n,m}$, and $\mu_{n,m}$ would be more accurate. For simplicity of presentation, however, we suppress this dependence whenever possible without ambiguity.

Expansion independence has a similar normative implication to existence of constant critical levels. In the presence of transitivity, existence of constant critical levels implies that for all $n, m \in \mathbb{N}$ and for all $k \in \mathbb{N}$ with $k > \max\{n, m\}$, there exists $\alpha \in \mathbb{R}$ such that, for all $u \in \mathbb{R}^n$ and for all $v \in \mathbb{R}^m$,

$$uRv \;\Leftrightarrow\; (u, \underbrace{\alpha, \ldots, \alpha}_{k-n})R(v, \underbrace{\alpha, \ldots, \alpha}_{k-m}).$$

Critical levels can be used to adjust population sizes without changing the relative ranking between the two distributions. We note that the axiom of existence of constant critical levels encompasses the entire domain, while expansion independence focuses on the subdomain associated with those who are below the threshold. In a generalized sufficientarian ordering, the critical level δ_H cannot be below the threshold of sufficiency. Thus, existence of constant critical levels cannot be applied to that subdomain and, therefore, an axiom such as expansion independence is required.

Finally, we introduce three axioms that capture a concern for distributional equity below, above, and across the threshold; see Pigou (1912) and Dalton (1920) for the principle of transfers in the context of income distributions. The first of these requires that a progressive transfer between two individuals below the threshold is socially beneficial. The second applies to transfers that take place above the threshold and merely demands that the post-transfer distribution be at least as good as the pre-transfer utility allocation. The reason why we only impose a weak variant of the axiom is that we want to allow the resulting class of orderings to include Crisp's (2003) proposal.

Principle of transfers below the threshold. For all $n \in \mathbb{N}$, for all $u, v \in \mathbb{R}^n$, for all $i, j \in \{1, \ldots, n\}$, and for all $\varepsilon > 0$, if $u_i = v_i + \varepsilon \leq v_j - \varepsilon = u_j < v_j < \theta$ and $u_k = v_k$ for all $k \in \{1, \ldots, n\} \setminus \{i, j\}$, then uPv.

Weak principle of transfers above and at the threshold. For all $n \in \mathbb{N}$, for all $u, v \in \mathbb{R}^n$, for all $i, j \in \{1, \ldots, n\}$, and for all $\varepsilon > 0$, if $\theta \leq v_i < u_i = v_i + \varepsilon \leq v_j - \varepsilon = u_j$ and $u_k = v_k$ for all $k \in \{1, \ldots, n\} \setminus \{i, j\}$, then uRv.

As is well-known, these transfer principles are responsible for the curvature properties of the function g. The principle of transfers below the threshold leads to the strict concavity of g below the threshold, whereas the concavity of g on $\mathbb{R}_{HE}$ follows from the weak principle of transfers above and at the threshold.

In a result that identifies an important subclass of the generalized critical-level sufficientarian orderings, we also employ the following transfer principle across the threshold that applies to situations in which the donor of a transfer drops below the threshold as a consequence of the associated utility loss.

Principle of transfers across the threshold. For all $n \in \mathbb{N}$, for all $u, v \in \mathbb{R}^n$, for all $i, j \in \{1, \ldots, n\}$, and for all $\varepsilon > 0$, if $u_i = v_i + \varepsilon \leq v_j - \varepsilon = u_j < \theta \leq v_j$ and $u_k = v_k$ for all $k \in \{1, \ldots, n\} \setminus \{i, j\}$, then uPv.

4 Axiomatic analysis

We begin by showing that a constant critical level must be greater than or equal to the threshold of sufficiency in the presence of absolute priority and strong Pareto.

Theorem 1. *If a sufficientarian ordering R satisfies absolute priority, strong Pareto, and existence of constant critical levels, then $\alpha \geq \theta$, where α is the constant critical level.*

Proof. By existence of constant critical levels, there exists $\alpha \in \mathbb{R}$ such that, for all $u \in \Omega$, $uI(u, \alpha)$. By way of contradiction, suppose that $\alpha < \theta$. Let $\beta \in (\alpha, \theta)$, and consider $v \in \Omega_L$ and $w \in \Omega_{HE}$. By strong Pareto, we obtain $(v, \beta)P(v, \alpha)$. Since α is a constant critical level, we have $(v, \alpha)Iv$. Since R is transitive, it follows that $(v, \beta)Pv$. Let $\gamma \in \mathbb{R}$ be such that $\gamma > \theta$. Note that $(v, \beta) \in \Omega_L$ and $(\gamma, w) \in$

Ω_{HE}. Since R satisfies absolute priority, $(v, \beta)Pv$ implies $(v, \beta, w)P(v, \gamma, w)$. This contradicts strong Pareto. ■

Blackorby, Bossert, and Donaldson (2005, Theorem 6.8) show that if an ordering R satisfies anonymity, strong Pareto, existence independence, and weak existence of critical levels, then there exists a constant critical level α that applies to all utility distributions in Ω. This observation immediately gives us the following corollary.

Corollary 1. *If a sufficientarian ordering R satisfies absolute priority, anonymity, strong Pareto, existence independence, and weak existence of critical levels, then there exists a constant critical level α such that $\alpha \geq \theta$.*

A generalized critical-level sufficientarian ordering satisfies strong Pareto if it is associated with a critical level α that is greater than or equal to the threshold θ. As a simple illustration, suppose that we have a two-person situation with utilities (u_1, u_2) such that $u_1 < \theta \leq u_2$. Now consider any distribution (v_1, v_2) such that $(v_1, v_2) > (u_1, u_2)$, and suppose that $v_1 < \theta \leq v_2$. It follows that

$$g(v_1) - \delta_L \geq g(u_1) - \delta_L \ \text{ and } \ g(v_2) - \delta_H \geq g(u_2) - \delta_H.$$

Because at least one of these inequalities is strict, (v_1, v_2) is better than (u_1, u_2). Suppose now that $\theta \leq v_1$ and $\theta \leq u_2$ so that the first individual is deemed to have enough. For the distribution (v_1, v_2), the aggregate utility of those below the threshold is zero, and the aggregate utility of those above or at the threshold is $g(v_1) + g(v_2) - 2\delta_H$. We note that $g(v_1) + g(v_2) - 2\delta_H$ can be less than $g(u_2) - \delta_H$ but it must be true that $g(u_1) - \delta_L < 0$ because $\delta_L \geq \sup_{a \in (\infty, \theta)} g(a)$ and u_1 is less than θ. Therefore, (v_1, v_2) is better than (u_1, u_2) according to a generalized critical-level sufficientarian ordering with a critical level that is greater than or equal to the threshold.

Our main result is a characterization of the class of generalized critical-level sufficientarian orderings with a critical level that is greater than or equal to the threshold level. There are ten axioms employed in this characterization, a feature that may be seen as a drawback because this is a 'large' number. Our first response to this criticism is that the number of axioms in and of itself does not reveal the interaction between them; for starters, they are independent, as established in the appendix. Moreover, the number of axioms is, to some extent, a matter of how the requisite conditions are grouped together, and our choice in that regard is motivated by a desire to pin down the exact role played by each of the requirements. For instance, it would be a perfectly reasonable alternative to combine the two continuity properties and the two transfer principles to reduce the number of axioms.

Theorem 2. *A sufficientarian ordering R satisfies absolute priority, anonymity, strong Pareto, existence independence, continuity below the threshold, continuity above and at the threshold, weak existence of critical levels, expansion independence,* *the principle of transfers below the threshold, and the weak principle of transfers above and at the threshold if and only if R is a generalized critical-level sufficientarian ordering with a critical level α that is greater than or equal to the threshold level θ.*

We establish three auxiliary results before presenting the proof of Theorem 2. The first of these shows that anonymity and existence independence together imply that any replications of two utility distributions must be ranked in the same way as the original distributions. Note that the original utility distributions are allowed to have different population sizes. This property appears in Zoli (2009), who labeled it strong population replication principle; see also Blackorby, Bossert, and Donaldson (2005).

Lemma 1. *If a sufficientarian ordering R satisfies anonymity and existence independence, then, for all $n, m, k \in \mathbb{N}$ with $k \geq 2$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,*

$$uRv \;\Leftrightarrow\; (\underbrace{u, \ldots, u}_{k})R(\underbrace{v, \ldots, v}_{k}).$$

Proof. Let $n, m, k \in \mathbb{N}$ with $k \geq 2$, $u \in \mathbb{R}^n$, and $v \in \mathbb{R}^m$. First, to show that

$$uRv \;\Rightarrow\; (\underbrace{u, \ldots, u}_{k})R(\underbrace{v, \ldots, v}_{k}),$$

suppose that uRv. Let $\ell \in \{0, 1, \ldots, k-1\}$. By existence independence, it follows that

$$(u, \underbrace{u, \ldots, u}_{k-\ell-1}, \underbrace{v, \ldots, v}_{\ell})R(v, \underbrace{u, \ldots, u}_{k-\ell-1}, \underbrace{v, \ldots, v}_{\ell}). \tag{1}$$

By anonymity and the transitivity of R, it follows from (1) that

$$(\underbrace{u, \ldots, u}_{k-\ell}, \underbrace{v, \ldots, v}_{\ell})R(\underbrace{u, \ldots, u}_{k-\ell-1}, \underbrace{v, \ldots, v}_{\ell+1}). \tag{2}$$

Since (2) holds for all $\ell \in \{0, 1, \ldots, k-1\}$, we obtain, by the transitivity of R, that

$$(\underbrace{u, \ldots, u}_{k})R(\underbrace{v, \ldots, v}_{k}). \tag{3}$$

To show that

$$(\underbrace{u, \ldots, u}_{k})R(\underbrace{v, \ldots, v}_{k}) \;\Rightarrow\; uRv,$$

suppose that

$$(\underbrace{u, \ldots, u}_{k})R(\underbrace{v, \ldots, v}_{k}).$$

By way of contradiction, we assume that uRv does not hold. Since R is complete, we have vPu. Using the argument employed to show (3), we obtain

$$(\underbrace{v,\ldots,v}_{k})P(\underbrace{u,\ldots,u}_{k}).$$

This is a contradiction. Thus, uRv must hold. ■

The following lemma shows that if an ordering R satisfies some of the axioms of the theorem statement, it ranks utility distributions at or above the threshold by applying critical-level generalized utilitarianism.

Lemma 2. *If a sufficientarian ordering R satisfies anonymity, strong Pareto, existence independence, continuity above and at the threshold, weak existence of critical levels, and the weak principle of transfers above and at the threshold, then there exists a continuous, increasing, and concave function $g_{HE}\colon \mathbb{R}_{HE} \to \mathbb{R}$ and $\delta_H \in \mathbb{R}$ with $g_{HE}(\theta) \le \delta_H < \sup_{a\in\mathbb{R}_{HE}} g(a)$ such that, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n_{HE}$, and for all $v \in \mathbb{R}^m_{HE}$,*

$$uRv \;\Leftrightarrow\; \sum_{i=1}^{n}[g_{HE}(u_i) - \delta_H] \ge \sum_{i=1}^{m}[g_{HE}(v_i) - \delta_H]. \tag{4}$$

Proof. Let R_{HE} denote the restriction of R to Ω_{HE}. By Corollary 1, there exists $\alpha \in \mathbb{R}_{HE}$ such that $uI(u,\alpha)$ for all $u \in \Omega$, which implies that $uI_{HE}(u,\alpha)$ for all $u \in \Omega_{HE}$. Furthermore, since R satisfies anonymity, strong Pareto, existence independence, and continuity above and at the threshold, the ordering R_{HE} on Ω_{HE} satisfies the properties corresponding to these axioms on Ω_{HE}. Applying Theorem 6.10 of Blackorby, Bossert, and Donaldson (2005), it follows that there exists a continuous and increasing function $g_{HE}\colon \mathbb{R}_{HE} \to \mathbb{R}$ such that, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n_{HE}$, and for all $v \in \mathbb{R}^m_{HE}$,

$$uR_{HE}v \;\Leftrightarrow\; \sum_{i=1}^{n}\left[g_{HE}(u_i) - g_{HE}(\alpha)\right] \ge \sum_{i=1}^{m}\left[g_{HE}(v_i) - g_{HE}(\alpha)\right].$$

By the weak principle of transfers above and at the threshold, g_{HE} is concave. Let $\delta_H = g_{HE}(\alpha)$. Since g_{HE} is increasing,

$$\sup_{a\in\mathbb{R}_{HE}} g(a) > \delta_H = g_{HE}(\alpha) \ge g_{HE}(\theta)$$

and the proof is complete. ■

Our final auxiliary result establishes an observation that is parallel to that of the previous lemma. It states that the ordering R ranks utility distributions below the threshold by applying a variant of critical-level generalized utilitarianism that utilizes δ_L.

Lemma 3. *If a sufficientarian ordering R satisfies anonymity, strong Pareto, existence independence, continuity below the threshold, expansion independence, and the principle of transfers below the threshold, then there exists an increasing and strictly concave function $g_L \colon \mathbb{R}_L \to \mathbb{R}$ and $\delta_L \in \mathbb{R}$ such that, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n_L$, and for all $v \in \mathbb{R}^m_L$,*

$$uRv \;\Leftrightarrow\; \sum_{i=1}^{n} [g_L(u_i) - \delta_L] \geq \sum_{i=1}^{m} [g_L(v_i) - \delta_L]. \tag{5}$$

Proof. The proof proceeds in five steps.

Step 1. We show that there exists an increasing and strictly concave (and, thus, continuous) function $g_L \colon \mathbb{R}_L \to \mathbb{R}$ such that, for all $n \in \mathbb{N}$ and for all $u, v \in \mathbb{R}^n_L$,

$$uRv \;\Leftrightarrow\; \sum_{i=1}^{n} g_L(u_i) \geq \sum_{i=1}^{n} g_L(v_i). \tag{6}$$

For all $n \in \mathbb{N}$, let R^n_L be the restriction of R to $\mathbb{R}^n_L$. Since R satisfies anonymity, strong Pareto, existence independence, and continuity below the threshold, R^n_L inherits these properties on $\mathbb{R}^n_L$. Thus, for all $n \geq 3$, there exists a continuous and increasing function $g^n_L \colon \mathbb{R}_L \to \mathbb{R}$ such that, for all $u, v \in \mathbb{R}^n_L$,

$$uR^n_L v \;\Leftrightarrow\; \sum_{i=1}^{n} g^n_L(u_i) \geq \sum_{i=1}^{n} g^n_L(v_i);$$

see Blackorby, Bossert, and Donaldson (2005, Theorem 4.7). Since g^n_L is unique up to a positive affine function and R satisfies existence independence, we may choose $g_L = g^n_L$ for all $n \geq 3$ (see Blackorby, Bossert, and Donaldson, 2005, Chapter 6). Thus, for all $n \geq 3$ and for all $u, v \in \mathbb{R}^n_L$,

$$uRv \;\Leftrightarrow\; \sum_{i=1}^{n} g_L(u_i) \geq \sum_{i=1}^{n} g_L(v_i).$$

By existence independence, this result extends to population sizes 1 and 2. Since R satisfies the principle of transfers below the threshold, g_L must be strictly concave.

Step 2. We show that, for all $n, m \in \mathbb{N}$ with $n \neq m$, there exists a unique $\delta^{n,m}_L \in \mathbb{R}$ such that, for all $u \in \mathbb{R}^n_L$ and for all $v \in \mathbb{R}^m_L$,

$$uRv \;\Leftrightarrow\; \sum_{i=1}^{n} [g_L(u_i) - \delta^{n,m}_L] \geq \sum_{i=1}^{m} [g_L(v_i) - \delta^{n,m}_L], \tag{7}$$

where $g_L \colon \mathbb{R}_L \to \mathbb{R}$ is an increasing and strictly concave function that satisfies (6).

Let $n, m \in \mathbb{N}$ with $n \neq m$. Since R satisfies expansion independence, there exist $k_{n,m} \in \mathbb{N}$ with $k_{n,m} > \max\{n, m\}$ and $\lambda_{n,m}, \mu_{n,m} \in \mathbb{R}_L$ such that, for all $u \in \mathbb{R}^n_L$ and for all $v \in \mathbb{R}^m_L$,

$$uRv \;\Leftrightarrow\; (u, \underbrace{\lambda_{n,m}, \ldots, \lambda_{n,m}}_{k_{n,m}-n})R(v, \underbrace{\mu_{n,m}, \ldots, \mu_{n,m}}_{k_{n,m}-m})$$

and

$$vRu \;\Leftrightarrow\; (v, \underbrace{\mu_{n,m}, \ldots, \mu_{n,m}}_{k_{n,m}-m})R(u, \underbrace{\lambda_{n,m}, \ldots, \lambda_{n,m}}_{k_{n,m}-n}).$$

Thus, it follows from Step 1 that, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n_L$, and for all $v \in \mathbb{R}^m_L$,

$$\begin{aligned} uRv \;&\Leftrightarrow\; \sum_{i=1}^{n} g_L(u_i) + (k_{n,m} - n)g_L(\lambda_{n,m}) \geq \sum_{i=1}^{m} g_L(v_i) + (k_{n,m} - m)g_L(\mu_{n,m}) \\ &\Leftrightarrow\; \sum_{i=1}^{n} g_L(u_i) + [(k_{n,m} - n)g_L(\lambda_{n,m}) - (k_{n,m} - m)g_L(\mu_n)] \geq \sum_{i=1}^{m} g_L(v_i) \end{aligned}$$

and

$$\begin{aligned} vRu \;&\Leftrightarrow\; \sum_{i=1}^{m} g_L(v_i) + (k_{n,m} - m)g_L(\mu_{n,m}) \geq \sum_{i=1}^{n} g_L(u_i) + (k_{n,m} - n)g_L(\lambda_{n,m}) \\ &\Leftrightarrow\; \sum_{i=1}^{m} g_L(v_i) \geq \sum_{i=1}^{n} g_L(u_i) + [(k_{n,m} - n)g_L(\lambda_{n,m}) - (k_{n,m} - m)g_L(\mu_{n,m})]. \end{aligned}$$

Given $n, m \in \mathbb{N}$ with $n \neq m$, we define $\delta^{n,m}_L \in \mathbb{R}$ by

$$\delta^{n,m}_L = \frac{(k_{n,m} - n)g_L(\lambda_{n,m}) - (k_{n,m} - m)g_L(\mu_{n,m})}{m - n}. \tag{8}$$

By definition, $\delta^{n,m}_L$ satisfies (7) for all $u \in \mathbb{R}^n_L$ and for all $v \in \mathbb{R}^m_L$.

To prove the uniqueness of $\delta^{n,m}_L$, we show that, for all $n, m \in \mathbb{N}$ with $n \neq m$, there exist $(a, \ldots, a) \in \mathbb{R}^n_L$ and $(b, \ldots, b) \in \mathbb{R}^m_L$ such that $(a, \ldots, a)I(b, \ldots, b)$. Let $n, m \in \mathbb{N}$ with $n \neq m$. Without loss of generality, suppose that $m > n$. Using $\delta^{n,m}_L$ given by (8), we define $\Delta^{n,m}$ by

$$\Delta^{n,m} = (m - n)\delta^{n,m}_L.$$

Since $\delta^{n,m}_L$ satisfies (7) for all $u \in \mathbb{R}^n_L$ and for all $v \in \mathbb{R}^m_L$, it follows that

$$(\underbrace{a, \ldots, a}_{n})I(\underbrace{b, \ldots, b}_{m}) \;\Leftrightarrow\; mg_L(b) - ng_L(a) = \Delta^{n,m}$$

for all $a, b \in \mathbb{R}_L$. We distinguish two cases.

First, suppose that $\Delta^{n,m} > 0$. Since g_L is increasing and strictly concave on $\mathbb{R}_L$, g_L is not bounded below. Thus, there exists $b \in \mathbb{R}_L$ such that

$$g_L(b) < 0.$$

Furthermore, since $m > n$, there exists $c \in (-\infty, b) \subset \mathbb{R}_L$ such that

$$g_L(c) < \frac{mg_L(b) - \Delta^{n,m}}{n} < \frac{m}{n} g_L(b) < g_L(b).$$

Because g_L is continuous, it follows from the intermediate value theorem that there exists $a \in (c, b) \subset \mathbb{R}_L$ such that

$$g_L(a) = \frac{mg_L(b) - \Delta^{n,m}}{n},$$

that is, $mg_L(b) - ng_L(a) = \Delta^{n,m}$.

Now suppose that $\Delta^{n,m} \leq 0$. We further distinguish two subcases.

First, suppose that there exists $c \in \mathbb{R}_L$ such that $g_L(c) = 0$. Let $a \in (c, \theta) \subset \mathbb{R}_L$. Since g_L is increasing, $g_L(a) > 0$. Then, it follows that

$$\frac{ng_L(a) + \Delta^{n,m}}{m} \leq \frac{n}{m} g_L(a) < g_L(a).$$

Since g_L is not bounded below, there exists $d \in (-\infty, a) \subset \mathbb{R}_L$ such that

$$g_L(d) < \frac{ng_L(a) + \Delta^{n,m}}{m}.$$

By the continuity of g_L, there exists $b \in (d, a) \subset \mathbb{R}_L$ such that

$$g_L(b) = \frac{ng_L(a) + \Delta^{n,m}}{m}.$$

In the second subcase, there is no $c \in \mathbb{R}_L$ such that $g_L(c) = 0$. Because g_L is increasing, $g_L(c) < 0$ for all $c \in \mathbb{R}_L$. Since g_L is not bounded below, there exists $b \in \mathbb{R}_L$ such that $mg_L(b) - ng_L(b) \leq g_L(b) < \Delta^{n,m}$. Furthermore, $\lim_{c \to -\infty} -g_L(c) = \infty$. Thus, there exists $d \in (-\infty, b) \subset \mathbb{R}_L$ such that

$$mg_L(b) - ng_L(b) < \Delta^{n,m} < mg_L(b) - ng_L(d).$$

Therefore, we obtain

$$g_L(b) > \frac{mg_L(b) - \Delta^{n,m}}{n} > g_L(d).$$

By the continuity of g_L, there exists $a \in (d, b) \subset \mathbb{R}_L$ such that

$$g_L(a) = \frac{mg_L(b) - \Delta^{n,m}}{n}.$$

To complete Step 2, let $n, m \in \mathbb{N}$ with $n \neq m$ and suppose, by way of contradiction, that there exists $\delta \in \mathbb{R}$ with $\delta \neq \delta_L^{n,m}$ that satisfies (7) for all $u \in \mathbb{R}_L^n$ and for all $v \in \mathbb{R}_L^m$. Without loss of generality, suppose that $\delta > \delta_L^{n,m}$ and $m > n$. Let $(a, \ldots, a) \in \mathbb{R}_L^n$ and $(b, \ldots, b) \in \mathbb{R}_L^m$ be such that $(a, \ldots, a)I(b, \ldots, b)$. Since $\delta_L^{n,m}$ satisfies (7) for all $u \in \mathbb{R}_L^n$ and for all $v \in \mathbb{R}_L^m$, we obtain

$$mg_L(b) - ng_L(a) = (m - n)\delta_L^{n,m}.$$

Since $(m - n)\delta_L^{n,m} < (m - n)\delta$ and δ satisfies (7) for all $u \in \mathbb{R}_L^n$ and for all $v \in \mathbb{R}_L^m$, we have

$$(\underbrace{b, \ldots, b}_{m})P(\underbrace{a, \ldots, a}_{n}).$$

This is a contradiction. Thus, only $\delta_L^{n,m}$ satisfies (7) for all $u \in \mathbb{R}_L^n$ and for all $v \in \mathbb{R}_L^m$.

Step 3. We show that there exists a unique $\delta_L \in \mathbb{R}$ such that, for all $n \in \mathbb{N}$, for all $u \in \mathbb{R}_L^n$, and for all $v \in \mathbb{R}_L^{n+1}$,

$$\begin{aligned} uRv \;&\Leftrightarrow\; \sum_{i=1}^{n}[g_L(u_i) - \delta_L] \geq \sum_{i=1}^{n+1}[g_L(v_i) - \delta_L] \quad \text{and} \\ vRu \;&\Leftrightarrow\; \sum_{i=1}^{n+1}[g_L(v_i) - \delta_L] \geq \sum_{i=1}^{n}[g_L(u_i) - \delta_L], \end{aligned} \tag{9}$$

where $g_L \colon \mathbb{R}_L \to \mathbb{R}$ is an increasing and strictly concave function that satisfies (6). That is, δ_L satisfies (5) in the lemma statement for any $n \in \mathbb{N}$ and $m = n + 1$.

For simplicity, we denote $\delta_L^{n,n+1}$ by δ_L^n for all $n \in \mathbb{N}$. We show that $\delta_L^n = \delta_L^{n+1}$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, $u \in \mathbb{R}_L^n$, $v \in \mathbb{R}_L^{n+1}$, and $a \in \mathbb{R}_L$. By existence independence, we obtain

$$\begin{aligned} uRv \;&\Leftrightarrow\; (u, a)R(v, a) \\ &\Leftrightarrow\; \sum_{i=1}^{n} g_L(u_i) + g_L(a) - (n + 1)\delta_L^{n+1} \geq \sum_{i=1}^{n+1} g_L(v_i) + g_L(a) - (n + 2)\delta_L^{n+1} \\ &\Leftrightarrow\; \sum_{i=1}^{n}[g_L(u_i) - \delta_L^{n+1}] \geq \sum_{i=1}^{n+1}[g_L(v_i) - \delta_L^{n+1}] \end{aligned}$$

and, analogously,

$$vRu \;\Leftrightarrow\; \sum_{i=1}^{n+1}[g_L(v_i) - \delta_L^{n+1}] \geq \sum_{i=1}^{n}[g_L(u_i) - \delta_L^{n+1}].$$

Thus, for each $n \in \mathbb{N}$, δ_L^{n+1} satisfies (9). From Step 2, it follows that $\delta_L^n = \delta_L^{n+1}$ for all $n \in \mathbb{N}$. Thus, $\delta_L = \delta^n$ satisfies (9).

Step 4. We show that, for all $n, m, \ell \in \mathbb{N}$ with $|m - n| = 1$, for all $u \in \mathbb{R}_L^{n\ell}$, and for all $v \in \mathbb{R}_L^{m\ell}$, δ_L satisfies (5).

Let $n, m, \ell \in \mathbb{N}$ and suppose, without loss of generality, that $m = n + 1$. From Lemma 1 and Step 2, it follows that, for all $u \in \mathbb{R}_L^n$ and for all $v \in \mathbb{R}_L^{n+1}$,

$$\begin{aligned} uRv &\Leftrightarrow (\underbrace{u, \ldots, u}_{\ell})R(\underbrace{v, \ldots, v}_{\ell}) \\ &\Leftrightarrow \ell \sum_{i=1}^{n} g_L(u_i) + \ell \delta_L^{n\ell,(n+1)\ell} \geq \ell \sum_{i=1}^{n+1} g_L(v_i) \\ &\Leftrightarrow \sum_{i=1}^{n} g_L(u_i) + \delta_L^{n\ell,(n+1)\ell} \geq \sum_{i=1}^{n+1} g_L(v_i). \end{aligned}$$

Thus, it follows from Step 3 that, for all $n, \ell \in \mathbb{N}$,

$$\delta_L^{n\ell,(n+1)\ell} = \delta_L.$$

Since $\delta_L^{n\ell,(n+1)\ell}$ satisfies (5) for all $u \in \mathbb{R}_L^{n\ell}$ and for all $v \in \mathbb{R}_L^{m\ell}$, Step 4 is complete.

Step 5. We complete the proof. Let $n, m \in \mathbb{N}$, $u \in \mathbb{R}_L^n$, and $v \in \mathbb{R}_L^m$. Without loss of generality, suppose that $m > n$. Let $\ell = m - n$. Define $h \in \mathbb{N}$ by

$$h = \min\{h' \in \mathbb{N} \mid h'\ell > n\}.$$

Let

$$(u, \underbrace{a, \ldots, a}_{h\ell - n}) \in \mathbb{R}_L^{h\ell} \quad \text{and} \quad (v, \underbrace{a, \ldots, a}_{h\ell - n}) \in \mathbb{R}^{(h+1)\ell}.$$

From existence independence and Step 4, it follows that

$$\begin{aligned} uRv &\Leftrightarrow (u, \underbrace{a, \ldots, a}_{h\ell - n})R(v, \underbrace{a, \ldots, a}_{h\ell - n}) \\ &\Leftrightarrow \sum_{i=1}^{n}[g_L(u_i) - \delta_L] + (h\ell - n)[g_L(a) - \delta_L] \geq \sum_{i=1}^{m}[g_L(v_i) - \delta_L] + (h\ell - n)[g_L(a) - \delta_L] \\ &\Leftrightarrow \sum_{i=1}^{n}[g_L(u_i) - \delta_L] \geq \sum_{i=1}^{m}[g_L(v_i) - \delta_L] \end{aligned}$$

and the proof is complete. ■

We are now in a position to prove our characterization result.

Proof of Theorem 2. 'If.' Suppose that R is a generalized critical-level sufficientarian ordering associated with a function $g\colon \mathbb{R} \to \mathbb{R}$ and parameters $\delta_L, \delta_H \in \mathbb{R}$, where g is increasing and strictly concave (thus continuous) on $\mathbb{R}_L$, continuous, increasing, and concave on $\mathbb{R}_{HE}$, and δ_L and δ_H are such that the inequalities $\delta_L \geq \sup_{a \in \mathbb{R}_L} g(a)$ and $\sup_{a \in \mathbb{R}_{HE}} > \delta_H \geq g(\theta)$ are satisfied. We show that R satisfies the axioms of the theorem.

To see that R satisfies absolute priority, let $n, m, \ell, r \in \mathbb{N}$, $u \in \mathbb{R}^n_L$, $v \in \mathbb{R}^m_L$, $w \in \mathbb{R}^\ell_{HE}$, and $s \in \mathbb{R}^r_{HE}$. To prove part (a) of the property, suppose that uPv. Since $E^n(u) \cup H^n(u) = E^m(v) \cup H^m(v) = \varnothing$, it follows that

$$\sum_{i \in E^n(u) \cup H^n(u)} [g(u_i) - \delta_H] = 0 = \sum_{i \in E^m(v) \cup H^m(v)} [g(v_i) - \delta_H].$$

By the definition of R, uPv implies

$$\sum_{i \in L^n(u)} [g(u_i) - \delta_L] > \sum_{i \in L^m(v)} [g(v_i) - \delta_L].$$

Let $\bar{u} = (u, w)$ and $\bar{v} = (v, s)$. Since $L^\ell(w) = L^r(s) = \varnothing$, it follows that $L^{n+\ell}(u, w) = L^n(u)$ and $L^{m+r}(v, s) = L^m(v)$. Thus,

$$\sum_{i \in L^{n+\ell}(u,w)} [g(\bar{u}_i) - \delta_L] = \sum_{i \in L^n(u)} [g(u_i) - \delta_L] > \sum_{i \in L^m(v)} [g(v_i) - \delta_L] = \sum_{i \in L^{m+r}(v,s)} [g(\bar{v}_i) - \delta_L].$$

By the definition of R, we obtain $(u, w)P(v, s)$. The proof of part (b) is analogous since $E^n(u) \cup H^n(u) = E^m(v) \cup H^m(v) = \varnothing$ and $L^\ell(w) = L^r(s) = \varnothing$.

That R satisfies anonymity follows immediately by definition.

We next show that R satisfies strong Pareto. Let $n \in \mathbb{N}$ and $u, v \in \mathbb{R}^n$, and suppose that $u > v$. Note that $L^n(v) \subseteq L^n(u)$.

First, suppose that $L^n(u) = L^n(v)$. Since g is increasing, we obtain

$$\sum_{i \in L^n(u)} [g(u_i) - \delta_L] > \sum_{i \in L^n(v)} [g(v_i) - \delta_L]$$

or

$$\sum_{i \in L^n(u)} [g(u_i) - \delta_L] = \sum_{i \in L^n(v)} [g(v_i) - \delta_L] \text{ and } \sum_{i \in E^n(u) \cup H^n(u)} [g(u_i) - \delta_H] > \sum_{i \in E^n(v) \cup H^n(v)} [g(v_i) - \delta_H].$$

In either case, we obtain uPv.

Next, suppose that $L^n(v) \subset L^n(u)$. Since $\delta_L \geq \sup_{a \in \mathbb{R}_L} g(a)$, it follows that

$$\sum_{i \in L^n(u)} [g(u_i) - \delta_L] > \sum_{i \in L^n(v)} [g(v_i) - \delta_L]$$

and, therefore, uPv.

Now we show that R satisfies existence independence. Let $n, m, \ell \in \mathbb{N}$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, and $w \in \mathbb{R}^\ell$. Let $\bar{u} = (u, w)$ and $\bar{v} = (v, w)$. It follows that

$$\sum_{i \in L^{n+\ell}(\bar{u})} [g(\bar{u}_i) - \delta_L] \geq \sum_{i \in L^{m+\ell}(\bar{v})} [g(\bar{v}_i) - \delta_L] \Leftrightarrow \sum_{i \in L^n(u)} [g(u_i) - \delta_L] \geq \sum_{i \in L^m(v)} [g(v_i) - \delta_L]$$

and

$$\sum_{i \in E^{n+\ell}(\bar{u}) \cup H^{n+\ell}(\bar{u})} [g(\bar{u}_i) - \delta_H] \geq \sum_{i \in E^{m+\ell}(\bar{v}) \cup H^{m+\ell}(\bar{v})} [g(\bar{v}_i) - \delta_H]$$
$$\Leftrightarrow \sum_{i \in E^n(u) \cup H^n(u)} [g(u_i) - \delta_H] \geq \sum_{i \in E^m(v) \cup H^m(v)} [g(v_i) - \delta_H].$$

Thus, by the definition of R,

$$uRv \Leftrightarrow (u, w)R(v, w).$$

Now we show that R satisfies continuity below the threshold. Let $n \in \mathbb{N}$ and $u \in \mathbb{R}^n_L$. To show that $\{v \in \mathbb{R}^n_L \mid vPu\}$ is open in $\mathbb{R}^n_L$, let $v^* \in \{v \in \mathbb{R}^n_L \mid vPu\}$. Since v^*Pu and $u, v^* \in \mathbb{R}^n_L$, it follows that

$$\sum_{i=1}^n g(v_i^*) > \sum_{i=1}^n g(u_i).$$

Define $\Delta \in \mathbb{R}_{++}$ by

$$\Delta = \sum_{i=1}^n g(v_i^*) - \sum_{i=1}^n g(u_i).$$

Let $b \in \mathbb{R}_L$ be such that $b = \min\{v_1^*, \ldots, v_n^*\}$. Since g is continuous and increasing on $\mathbb{R}_L$ and is not bounded below, there exists $\varepsilon \in \mathbb{R}_{++}$ such that $b + \varepsilon < \theta$ and

$$g(b) - g(b - \varepsilon) < \frac{\Delta}{n}.$$

Since g is strictly concave, it follows that, for all $i \in \{1, \ldots, n\}$,

$$g(v_i^*) - g(v_i^* - \varepsilon) \leq g(b) - g(b - \varepsilon).$$

Thus, we obtain

$$\sum_{i=1}^n g(v_i^*) - \sum_{i=1}^n g(v_i^* - \varepsilon) \leq n[g(b) - g(b - \varepsilon)] < \Delta. \tag{10}$$

Let $B_\varepsilon(v^*) \subset \mathbb{R}^n_L$ be the open ball with radius ε and center v^*. Note that, for all $v \in B_\varepsilon(v^*)$ and for all $i \in \{1, \ldots, n\}$,

$$|v_i^* - v_i| < \varepsilon.$$

Thus, by (10), it follows that, for all $v \in B_\varepsilon(v^*)$,

$$\sum_{i=1}^n g(v_i) - \sum_{i=1}^n g(u_i) = \sum_{i=1}^n g(v_i^*) - \sum_{i=1}^n g(u_i) - \left[\sum_{i=1}^n g(v_i^*) - \sum_{i=1}^n g(v_i)\right] > 0$$

and, thus,

$$B_\varepsilon(v^*) \subseteq \{v \in \mathbb{R}^n_L \mid vPu\}.$$

Therefore, $\{v \in \mathbb{R}^n_L \mid vPu\}$ is open in $\mathbb{R}^n_L$. The proof that $\{v \in \mathbb{R}^n_L \mid uPv\}$ is open in $\mathbb{R}^n_L$ is analogous.

To prove that R satisfies continuity above and at the threshold, let $n \in \mathbb{N}$ and $u \in \mathbb{R}^n_{HE}$. To show that $\{v \in \mathbb{R}^n_{HE} \mid vRu\}$ is closed in $\mathbb{R}^n_{HE}$, let $\langle v^t \rangle_{t\in\mathbb{N}}$ be a sequence in $\{v \in \mathbb{R}^n_{HE} \mid vRu\}$ and assume that $\langle v^t \rangle_{t\in\mathbb{N}}$ converges to $v^* \in \mathbb{R}^n_{HE}$. By way of contradiction, suppose that $v^* \notin \{v \in \mathbb{R}^n_{HE} \mid vRu\}$. Because R is complete, it follows that uPv^*. Since $v^*, u \in \mathbb{R}^n_{HE}$, we obtain

$$\sum_{i=1}^n g(u_i) > \sum_{i=1}^n g(v_i^*).$$

We define $\Delta \in \mathbb{R}_{++}$ by

$$\Delta = \sum_{i=1}^n g(u_i) - \sum_{i=1}^n g(v_i^*).$$

Since g is continuous on $\mathbb{R}^n_{HE}$ and $\langle v^t \rangle_{t\in\mathbb{N}}$ converges to $v^* \in \mathbb{R}^n_{HE}$, there exists $t^* \in \mathbb{N}$ such that, for all $t \geq t^*$,

$$\left|\sum_{i=1}^n g(v_i^*) - \sum_{i=1}^n g(v_i^t)\right| < \Delta,$$

which means that uPv^t for all $t \geq t^*$. This is a contradiction since v^tRu for all $t \in \mathbb{N}$. Thus, $v^* \in \{v \in \mathbb{R}^n_{HE} \mid vRu\}$, that is, $\{v \in \mathbb{R}^n_{HE} \mid vRu\}$ is closed in $\mathbb{R}^n_{HE}$. The proof that $\{v \in \mathbb{R}^n_{HE} \mid uRv\}$ is closed in $\mathbb{R}^n_{HE}$ is analogous.

We show that R satisfies weak existence of critical levels. Because $\sup_{a\in\mathbb{R}_{HE}} g(a) > \delta_H \geq g(\theta)$ and g is increasing and continuous on $\mathbb{R}_{HE}$, there exists $\alpha \in \mathbb{R}_{HE}$ such that $\delta_H = g(\alpha)$. Let $n \in \mathbb{N}$, $u \in \mathbb{R}^n$, and $v = (u, \alpha)$. We obtain

$$\sum_{i\in L^n(u)} [g(u_i) - \delta_L] = \sum_{i\in L^{n+1}(v)} [g(v_i) - \delta_L]$$

and

$$\sum_{i\in E^n(u)\cup H^n(u)} [g(u_i) - \delta_H] = \sum_{i\in E^n(u)\cup H^n(u)} [g(u_i) - \delta_H] + g(\alpha) - \delta_H = \sum_{i\in E^{n+1}(v)\cup H^{n+1}(v)} [g(v_i) - \delta_H].$$

Thus, $uI(u, \alpha)$ follows.

We now show that R satisfies expansion independence. Let $n, m \in \mathbb{N}$. If $n = m$, then the requisite condition is satisfied for any $k > \max\{m, n\}$ and $\lambda, \mu \in \mathbb{R}_L$ such that $\lambda = \mu$. Thus, we consider the case in which $n \neq m$. Without loss of generality, suppose that $m > n$. Let $k = m + 1$. We show that there exist $\lambda, \mu \in \mathbb{R}_L$ such that

$$\delta_L = \frac{(k-n)g(\lambda) - (k-m)g(\mu)}{m-n} = \frac{(m-n+1)g(\lambda) - g(\mu)}{m-n}. \tag{11}$$

Since g is increasing on $\mathbb{R}_L$ and $\delta_L \geq \sup_{a \in \mathbb{R}_L} g(a)$, $g(a) < \delta_L$ for all $a \in \mathbb{R}_L$. Furthermore, g is not bounded below because of strict concavity. Thus, there exist $\lambda, a \in \mathbb{R}_L$ with $\lambda > a$ such that

$$g(a) < g(\lambda) - (m-n)[\delta_L - g(\lambda)] < g(\lambda) < 0.$$

The function g is continuous on $\mathbb{R}_L$ as a consequence of strict concavity. Thus, it follows from the intermediate value theorem that there exists $\mu \in (a, \lambda) \subset \mathbb{R}_L$ such that

$$g(\mu) = g(\lambda) - (m-n)[\delta_L - g(\lambda)],$$

that is,

$$\delta_L = \frac{(m-n+1)g(\lambda) - g(\mu)}{m-n}.$$

Let $u \in \mathbb{R}^n_L$ and $v \in \mathbb{R}^m_L$. We define $\bar{u}, \bar{v} \in \mathbb{R}^{m+1}$ by

$$\bar{u} = (u, \underbrace{\lambda, \ldots, \lambda}_{m-n+1}) \quad \text{and} \quad \bar{v} = (v, \mu).$$

We show that

$$uRv \;\Leftrightarrow\; \bar{u}R\bar{v}.$$

Since λ and μ satisfy (11), we obtain

$$\begin{aligned}
uRv \;&\Leftrightarrow\; \sum_{i \in L^n(u)} [g(u_i) - \delta_L] \geq \sum_{i \in L^m(v)} [g(v_i) - \delta_L] \\
&\Leftrightarrow\; \sum_{i \in L^n(u)} g(u_i) + (m-n)\delta_L \geq \sum_{i \in L^m(v)} g(v_i) \\
&\Leftrightarrow\; \sum_{i \in L^n(u)} g(u_i) + (m-n+1)g(\lambda) - g(\mu) \geq \sum_{i \in L^m(v)} g(v_i) \\
&\Leftrightarrow\; \sum_{i \in L^n(u)} g(u_i) + (m-n+1)g(\lambda) \geq \sum_{i \in L^m(v)} g(v_i) + g(\mu) \\
&\Leftrightarrow\; \sum_{i \in L^{m+1}(\bar{u})} g(\bar{u}_i) \geq \sum_{i \in L^{m+1}(\bar{v})} g(\bar{v}_i) \\
&\Leftrightarrow\; \bar{u}R\bar{v}.
\end{aligned}$$

The proof that vRu if and only if $\bar{v}R\bar{u}$ is analogous.

Finally, R satisfies the principle of transfers below the threshold and the weak principle of transfers above and at the threshold because g is strictly concave on $\mathbb{R}_L$ and concave on $\mathbb{R}_{HE}$.

'Only if.' Suppose that R satisfies the axioms of the theorem statement. From Lemma 2, there exist a continuous, increasing, and concave function $g_{HE}\colon \mathbb{R}_{HE} \to \mathbb{R}$ and $\delta_H \in \mathbb{R}$ with $\delta_H \geq g_{HE}(\theta)$ such that (4) is satisfied for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n_{HE}$, and for all $v \in \mathbb{R}^m_{HE}$. Furthermore, from Lemma 3, there exist an increasing and strictly concave (and thus continuous) function $g_L\colon \mathbb{R}_L \to \mathbb{R}$ and $\delta_L \in \mathbb{R}$ such that (5) is satisfied for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n_L$, and for all $v \in \mathbb{R}^m_L$.

Using the functions g_{HE} and g_L, we define the function $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(a) = \begin{cases} g_L(a) & \text{if } a \in \mathbb{R}_L, \\ g_{HE}(a) & \text{if } a \in \mathbb{R}_{HE}. \end{cases}$$

Because g inherits the corresponding properties of g_L and g_{HE}, g is increasing and strictly concave (and therefore continuous) on $\mathbb{R}_L$, and continuous, increasing, and concave on $\mathbb{R}_{HE}$.

We show that R is the generalized critical-level sufficientarian ordering associated with g, δ_H, and δ_L. Let $n, m \in \mathbb{N}$, $u \in \mathbb{R}^n$, and $v \in \mathbb{R}^m$. First, suppose that

$$\sum_{i \in L^n(u)} [g(u_i) - \delta_L] > \sum_{i \in L^m(v)} [g(v_i) - \delta_L] \tag{12}$$

or

$$\begin{aligned} \sum_{i \in L^n(u)} [g(u_i) - \delta_L] &= \sum_{i \in L^m(v)} [g(v_i) - \delta_L] \quad \text{and} \\ \sum_{i \in H^n(u) \cup E^n(u)} [g(u_i) - \delta_H] &> \sum_{i \in H^m(v) \cup E^m(v)} [g(v_i) - \delta_H]. \end{aligned} \tag{13}$$

We show that uPv follows in either case.

First, suppose that (12) is true. By (5), it follows that

$$(u_i)_{i \in L^n(u)} \, P \, (v_i)_{i \in L^m(v)}.$$

Thus, by part (a) of absolute priority, we obtain

$$\left((u_i)_{i \in L^n(u)}, (u_i)_{i \in E^n(u) \cup H^n(u)}\right) P \left((v_i)_{i \in L^m(v)}, (v_i)_{i \in E^m(v) \cup H^m(v)}\right).$$

Since R satisfies anonymity, we obtain

$$\left((u_i)_{i \in L^n(u)}, (u_i)_{i \in E^n(u) \cup H^n(u)}\right) Iu \quad \text{and} \quad \left((v_i)_{i \in L^m(v)}, (v_i)_{i \in E^m(v) \cup H^m(v)}\right) Iv.$$

Thus, by the transitivity of R, it follows that uPv.

Now suppose that (13) is true. Since g and δ_L satisfy (5), it follows that

$$(u_i)_{i\in L^n(u)}\, I\, (v_i)_{i\in L^m(v)}.$$

Furthermore, by (4), we obtain

$$(u_i)_{i\in E^n(u)\cup H^n(u)}\, P\, (v_i)_{i\in E^m(v)\cup H^m(v)}.$$

By part (b) of absolute priority, it follows that

$$\left((u_i)_{i\in L^n(u)}, (u_i)_{i\in E^n(u)\cup H^n(u)}\right) P \left((v_i)_{i\in L^m(v)}, (v_i)_{i\in E^m(v)\cup H^m(v)}\right).$$

Since R is transitive and satisfies anonymity, we obtain uPv.

Next, suppose that

$$\sum_{i\in L^n(u)} [g(u_i) - \delta_L] = \sum_{i\in L^m(v)} [g(v_i) - \delta_L]$$

and

$$\sum_{i\in E^n(u)\cup H^n(u)} [g(u_i) - \delta_H] = \sum_{i\in E^m(v)\cup H^m(v)} [g(v_i) - \delta_H].$$

We show that uIv. By (5),

$$(u_i)_{i\in L^n(u)}\, I\, (v_i)_{i\in L^m(v)}.$$

Analogously, by (4),

$$(u_i)_{i\in E^n(u)\cup H^n(u)}\, I\, (v_i)_{i\in E^m(v)\cup H^m(v)}.$$

By part (b) of absolute priority, we obtain

$$\left((u_i)_{i\in L^n(u)}, (u_i)_{i\in E^n(u)\cup H^n(u)}\right) I \left((v_i)_{i\in L^m(v)}, (v_i)_{i\in E^m(v)\cup H^m(v)}\right).$$

Because R is transitive and satisfies anonymity, it follows that uIv.

Since R is complete, the above argument implies that R is the generalized critical-level sufficientarian ordering associated with g, δ_H, and δ_L. To complete the proof, we show that $\delta_L \geq \sup_{a\in\mathbb{R}_L} g_L(a)$. By way of contradiction, suppose that $\delta_L < \sup_{a\in\mathbb{R}_L} g_L(a)$. Since g is continuous and increasing on $\mathbb{R}_L$, there exists $c \in \mathbb{R}_L$ such that $g_L(c) = \delta_L$. Let $a \in \mathbb{R}_{HE}$ and $b \in \mathbb{R}_L$ be such that $c < b < \theta < a$. Note that $g(b) > g(c) = \delta_L$. We define $u, v \in \mathbb{R}^3$ by

$$u = (b, b, a) \text{ and } v = (b, a, a).$$

We obtain

$$\sum_{i\in L^3(u)} [g(u_i) - \delta_L] = 2[g(b) - \delta_L] > g(b) - \delta_L = \sum_{i\in L^3(v)} [g(v_i) - \delta_L].$$

Since R is the generalized critical-level sufficientarian ordering associated with g, δ_H, and δ_L, we obtain uPv. However, this is a contradiction because R satisfies strong Pareto. Thus, $\delta_L \geq \sup_{a \in \mathbb{R}_L} g_L(a)$. ■

We now examine some distributional implications of the generalized critical-level sufficientarian orderings. Assume that a society is in possession of a sufficiently large amount of a resource that allows everyone to be above or at the sufficiency threshold. In the absence of population change, the optimal distribution with respect to a generalized critical-level sufficientarian ordering is identical to the solution of maximizing the total sum of transformed utility levels $g(u_i)$ subject to the constraints that requires everyone's utility be at least as large as the threshold of sufficiency. This implies that generalized critical-level sufficientarian orderings have the same distributional implication as the (generalized) utilitarian orderings with a floor constraint, a problem that has been examined extensively in experiments; see, for instance, Frohlich and Oppenheimer (1992), Faravelli (2007), and Gaertner and Schokkaert (2012). In these contributions, subjects are asked to choose what they consider the normatively most attractive option among a range of social orderings. According to Frohlich and Oppenheimer (1992), the utilitarian orderings with a floor constraint receive more support than each of the maximin and utilitarian orderings. Our axiomatic analysis provides a theoretical foundation of utilitarianism with a floor.

One difficulty of utilitarianism with a floor is that it does not provide rankings when there is not enough of the resource to ensure that everyone reaches the threshold. An obvious advantage of our sufficientarian ordering is that it is capable of providing distributional judgments in these cases as well. Clearly, the distribution favored by our orderings is crucially dependent on δ_L. If the value of this parameter approaches infinity, the resulting generalized critical-level sufficientarian ordering approaches Frankfurt's proposal as the primary criterion. However, any value of δ_L above $\sup_{a \in \mathbb{R}_L} g(a)$ leads to a violation of the principle of transfers across the threshold. To identify the requisite subclass of our orderings, we add this principle to our list of axioms to obtain the following result.

Theorem 3. *A sufficientarian ordering R satisfies absolute priority, anonymity, strong Pareto, existence independence, continuity below the threshold, continuity above and at the threshold, weak existence of critical levels, expansion independence, the principle of transfers below the threshold, the weak principle of transfers above and at the threshold, and the transfer principle across the threshold if and only if R is a generalized critical-level sufficientarian ordering such that $\delta_L = \sup_{a \in \mathbb{R}_L} g(a)$ with a critical level α that is greater than or equal to the threshold level θ.*

Proof. 'If.' Let R be a generalized critical-level sufficientarian ordering such that $\delta_L = \sup_{a \in \mathbb{R}_L} g(a)$. All axioms other than the principle of transfers across the threshold follow from Theorem 2. To prove that the remaining property is satisfied, suppose that $n \in \mathbb{N}$, $u, v \in \mathbb{R}^n$, for all $i, j \in \{1, \ldots, n\}$, and $\varepsilon > 0$ are such that $u_i = v_i + \varepsilon \leq v_j - \varepsilon = u_j < \theta \leq v_j$ and $u_k = v_k$ for all $k \in \{1, \ldots, n\} \setminus \{i, j\}$. By definition of R, we have to show that uPv, that is,

$$g(v_i + \varepsilon) - \sup_{a \in \mathbb{R}_L} g(a) + g(v_j - \varepsilon) - \sup_{a \in \mathbb{R}_L} g(a) > g(v_i) - \sup_{a \in \mathbb{R}_L} g(a)$$

or, equivalently,

$$\frac{1}{2}[g(v_i + \varepsilon) + g(v_j - \varepsilon)] > \frac{1}{2}[g(v_i) + \sup_{a \in \mathbb{R}_L} g(a)]. \tag{14}$$

Define the function $\bar{g} \colon \mathbb{R}_L \cup \{\theta\} \to \mathbb{R}$ by letting $\bar{g}(a) = g(a)$ for all $a \in \mathbb{R}_L$ and $\bar{g}(\theta) = \sup_{a \in \mathbb{R}_L} g(a)$. Because g is strictly concave, so is $\bar{g}$. Thus, (14) is equivalent to

$$\frac{1}{2}[\bar{g}(v_i + \varepsilon) + \bar{g}(v_j - \varepsilon)] > \frac{1}{2}[\bar{g}(v_i) + \bar{g}(\theta)]$$

which follows from the strict concavity of $\bar{g}$ because $v_i < v_i + \varepsilon \leq v_j - \varepsilon < \theta$.

'Only if.' Clearly, R must be a generalized critical-level sufficientarian ordering by virtue of Theorem 2. By way of contradiction, suppose that $\delta_L > \sup_{a \in \mathbb{R}_L} g(a)$. Consider a two-person distribution $(\theta, \theta - t)$, where $t > 0$ and a sufficiently small transfer from individual 1 to individual 2 in the amount of $\varepsilon > 0$. Then, the resulting distribution is $(\theta - \varepsilon, \theta - t + \varepsilon)$. We note that

$$g(\theta - t) - \delta_L > g(\theta - \varepsilon) + g(\theta - t + \varepsilon) - 2\delta_L,$$

provided that ε is sufficiently small. This implies $(\theta, \theta - t)P(\theta - \varepsilon, \theta - t + \varepsilon)$, a violation of the principle of transfers across the threshold. ■

Thus, if the principle of transfers across the threshold is added to our list of axioms, the resulting orderings are such that there exists a function $g \in G$ and $\delta_H \in \mathbb{R}$ with $g(\theta) \leq \delta_H < \sup_{a \in \mathbb{R}_{HE}} g(a)$ such that, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$, uRv if and only if

$$\sum_{i \in L^n(u)} \left[g(u_i) - \sup_{a \in \mathbb{R}_L} g(a)\right] > \sum_{i \in L^m(v)} \left[g(v_i) - \sup_{a \in \mathbb{R}_L} g(a)\right]$$

or

$$\sum_{i \in L^n(u)} \left[g(u_i) - \sup_{a \in \mathbb{R}_L} g(a)\right] = \sum_{i \in L^m(v)} \left[g(v_i) - \sup_{a \in \mathbb{R}_L} g(a)\right]$$

and

$$\sum_{i \in E^n(u) \cup H^n(u)} [g(u_i) - \delta_H] \geq \sum_{i \in E^m(v) \cup H^m(v)} [g(v_i) - \delta_H].$$

5 Resolving a population-ethics dilemma

As noted in Section 2, a generalized critical-level sufficientarian ordering applies, as a tie-breaking device, a critical-level generalized-utilitarian ordering to the lifetime utilities of those at or above the threshold. In the context of population ethics, it is known that critical-level generalized utilitarianism cannot escape a well-known population-ethics dilemma: none of the members of this class allows us to avoid both Parfit's (1976, 1982, 1984) repugnant conclusion and Arrhenius's (2000) sadistic conclusion. As we illustrate later, this population-ethics dilemma also applies to other well-established utilitarian population principles. In contrast to these difficulties, generalized critical-level sufficientarianism has a subclass that allows us to avoid the dilemma.

An ordering R implies the repugnant conclusion if, for all $n \in \mathbb{N}$, for all $\xi \in \mathbb{R}_{++}$, and for all $\varepsilon \in (0, \xi)$, there exists $m \in \mathbb{N}$ with $m > n$ such that

$$(\underbrace{\varepsilon, \ldots, \varepsilon}_{m}) P (\underbrace{\xi, \ldots, \xi}_{n}).$$

According to a principle that implies the repugnant conclusion, population size can always be substituted for quality of life—any population with an arbitrarily high common level of lifetime well-being is declared inferior to some larger population in which everyone's utility is positive but arbitrarily close to neutrality. Critical-level generalized utilitarianism implies the repugnant conclusion if and only if the critical level α is non-positive. A prominent example of a class of principles that imply the repugnant conclusion is given by total generalized utilitarianism, which is the subclass of the critical-level generalized-utilitarian principles that results from setting the critical level equal to zero—the level of utility that represents a neutral life; see Parfit (1976, 1982, 1984).

The axiom that requires an ordering R to avoid the repugnant conclusion is defined as the negation of the repugnant conclusion.

Avoidance of the repugnant conclusion. There exist $n \in \mathbb{N}$, $\xi \in \mathbb{R}_{++}$, and $\varepsilon \in (0, \xi)$ such that, for all $m \in \mathbb{N}$ with $m > n$,

$$(\underbrace{\xi, \ldots, \xi}_{n}) R (\underbrace{\varepsilon, \ldots, \varepsilon}_{m}).$$

To define the sadistic conclusion and the axiom that requires avoiding it, we need additional notation. Let Ω_{++} and Ω_{--} denote the sets of all positive and all negative distributions, that is,

$$\Omega_{++} = \bigcup_{n \in \mathbb{N}} \mathbb{R}^n_{++} \ \text{ and } \ \Omega_{--} = \bigcup_{n \in \mathbb{N}} \mathbb{R}^n_{--}.$$

An ordering R implies the sadistic conclusion (Arrhenius, 2000) if there exist $u \in \Omega$, $v \in \Omega_{++}$, and $w \in \Omega_{--}$ such that $(u, w)P(u, v)$. According to this conclusion, it is possible that adding a number of people with equal negative utilities to an existing population is better than adding a (possibly different) number of people with equal positive utilities to the same existing population. Critical-level generalized utilitarianism implies the sadistic conclusion if and only if the critical level α is positive.

Again, the axiom that requires an ordering R to avoid the sadistic conclusion is defined as the negation of the sadistic conclusion.

Avoidance of the sadistic conclusion. For all $u \in \Omega$, for all $v \in \Omega_{++}$, and for all $w \in \Omega_{--}$, $(u, v)R(u, w)$.

The following theorem identifies the members of our class that avoid the repugnant conclusion and the sadistic conclusion—and, notably, those that avoid both of them.

Theorem 4. *Suppose that R is a generalized critical-level sufficientarian ordering.*

(a) R satisfies avoidance of the repugnant conclusion if and only if (i) $\theta > 0$ or (ii) $\delta_H > g(0)$.

(b) R satisfies avoidance of the sadistic conclusion if and only if (i) $\theta = 0$ or (ii) $\theta < 0$ and $\delta_H = g(0)$.

(c) R satisfies avoidance of the repugnant conclusion and avoidance of the sadistic conclusion if and only if $\theta = 0$ and $\delta_H > g(0)$.

Proof. (a) 'If.' First, suppose that $\theta > 0$. Let $n, m \in \mathbb{N}$ with $m > n$ and $\varepsilon \in (0, \theta)$. Define two distributions u and v by

$$u = (\underbrace{\theta, \ldots, \theta}_{n}) \text{ and } v = (\underbrace{\varepsilon, \ldots, \varepsilon}_{m}).$$

Note that $L^n(u) = \varnothing \neq L^m(v)$. It follows that

$$\sum_{i \in L^n(u)} [g(u_i) - \delta_L] = 0 > \sum_{i \in L^m(v)} [g(v_i) - \delta_L],$$

and we obtain uPv. Thus, R satisfies avoidance of the repugnant conclusion.

Next, suppose that $\delta_H > g(0)$. Since R satisfies avoidance of the repugnant conclusion if $\theta > 0$, we assume that $\theta \leq 0$. Since g is continuous and increasing on $\mathbb{R}_{HE}$ and $g(\theta) \leq \delta_H < \sup_{a \in \mathbb{R}_{HE}} g(a)$, there exists $\alpha \in \mathbb{R}_{HE}$ such that $g(\alpha) = \delta_H$. Since $\theta \leq 0$ and $g(0) < \delta_H = g(\alpha)$, we obtain $\alpha > 0$ because g is increasing on $\mathbb{R}_{HE}$. To show that R avoids the repugnant conclusion, let $\xi, \varepsilon \in \mathbb{R}$ be such that

$$\xi > \alpha > \varepsilon > 0 \geq \theta.$$

Let $n, m \in \mathbb{N}$ with $m > n$, and define the distributions u and v by

$$u = (\underbrace{\xi, \dots, \xi}_{n}) \text{ and } v = (\underbrace{\varepsilon, \dots, \varepsilon}_{m}).$$

Since $L^n(u) = \varnothing = L^m(v)$ and $g(\alpha) = \delta_H$, we obtain

$$\sum_{i \in L^n(u)} [g(u_i) - \delta_L] = 0 = \sum_{i \in L^m(v)} [g(v_i) - \delta_L]$$

and

$$\sum_{i \in E^n(u) \cup H^n(u)} [g(u_i) - \delta_H] > 0 > \sum_{i \in E^m(v) \cup H^m(v)} [g(v_i) - \delta_H].$$

This implies uPv. Thus, R satisfies avoidance of the repugnant conclusion.

'Only if.' By way of contraposition, suppose that $\theta \leq 0$ and $\delta_H \leq g(0)$. Let $n, m \in \mathbb{N}$ with $m > n$, $\xi \in \mathbb{R}_{++}$, and $\varepsilon \in (0, \xi)$. We define u and v by

$$u = (\underbrace{\xi, \dots, \xi}_{n}) \text{ and } v = (\underbrace{\varepsilon, \dots, \varepsilon}_{m}).$$

Since $\theta \leq 0$, it follows that $L^n(u) = \varnothing = L^m(v)$. Furthermore, since g is increasing on $\mathbb{R}_H E$, $g(\xi) > g(\varepsilon) > g(0) \geq \delta_H$. If m is sufficiently large, we obtain

$$0 < \sum_{i \in E^n(u) \cup H^n(u)} [g(u_i) - \delta_H] < \sum_{i \in E^m(v) \cup H^m(v)} [g(v_i) - \delta_H].$$

Thus, vPu if m is sufficiently large. This means that R implies the repugnant conclusion.

(b) 'If.' Let $u \in \Omega$, $v \in \Omega_{++}$, and $w \in \Omega_{--}$. Then there exist $n, m \in \mathbb{N}$ such that $v \in \mathbb{R}^n_{++}$ and $w \in \mathbb{R}^m_{--}$. Since R is generalized critical-level sufficientarian,

$$[(u, v)R(u, w) \Leftrightarrow vRw] \text{ and } [(u, w)R(u, v) \Leftrightarrow wRv].$$

First, suppose that $\theta = 0$. It follows that

$$\sum_{i \in L^m(w)} [g(w_i) - \delta_L] < 0 = \sum_{i \in L^n(v)} [g(w_i) - \delta_L].$$

Thus, we obtain vPw, which implies that $(u, v)P(u, w)$.

Next, suppose that $\theta < 0$ and $\delta_H = g(0)$, that is, the corresponding critical level α is equal to zero. We distinguish two cases. First, assume that there exists $i \in \{1, \dots, m\}$ such that $w_i < \theta$. This implies $L^m(w) \neq \varnothing = L^n(v)$, and we obtain

$$\sum_{i \in L^m(w)} [g(w_i) - \delta_L] < 0 = \sum_{i \in L^n(v)} [g(w_i) - \delta_L],$$

which implies $(u,v)P(u,w)$.

Now suppose that there is no $i \in \{1,\ldots,m\}$ such that $w_i < \theta$. Thus, $L^m(w) = \varnothing = L^n(v)$. Since $\delta_H = g(0)$, it follows that

$$\sum_{L^m(w)} [g(w_i) - \delta_L] = 0 = \sum_{L^n(v)} [g(w_i) - \delta_L]$$

and

$$\sum_{i \in E^m(w) \cup H^m(w)} [g(w_i) - \delta_H] < 0 < \sum_{i \in E^n(v) \cup H^n(v)} [g(w_i) - \delta_H].$$

Thus, we obtain $(u,v)P(u,w)$.

'Only if.' Suppose first that $\theta > 0$. Let a, b be such that $\theta > a > 0 > b$. Furthermore, let $n \in \mathbb{N}$ be such that

$$n[g(a) - \delta_L] < g(b) - \delta_L < 0.$$

Now define $v = (a,\ldots,a) \in \mathbb{R}^n_{++}$ and $w = (b) \in \mathbb{R}_{--}$. Consider any $u \in \Omega_{HE}$. We have

$$(u,w)P(u,v).$$

This means that R implies the sadistic conclusion.

Next, suppose that $\theta < 0$ and that the corresponding critical level α is less than zero. Let a, b be such that $a > 0 > b > \alpha$. Furthermore, let $n \in \mathbb{N}$ be such that

$$0 < g(a) - \delta_H < n[g(b) - \delta_H].$$

In analogy to the case $\theta > 0$, it follows that the sadistic conclusion is implied.

Finally, suppose that the corresponding critical level α is greater than zero. Let a, b be such that $\alpha > a > 0 > b > \theta$, and consider $n \in \mathbb{N}$ such that

$$n[g(a) - \delta_H] < [g(b) - \delta_H] < 0.$$

As in the previous case, the sadistic conclusion is implied. Thus, $\theta = 0$ or $\theta < 0$ and $\alpha = 0$.

Part (c) follows immediately from combining (a) and (b). ■

The possibility of avoiding both the repugnant conclusion and the sadistic conclusion is in stark contrast to two well-known utilitarian population principles other than critical-level generalized utilitarianism. Ng's (1986) number-dampened utilitarianism and its generalizations presented by Blackorby, Bossert, and Donaldson (2005) and Asheim and Zuber (2014) cannot avoid both the repugnant and sadistic conclusions. Its most general form examined in Asheim and Zuber (2014) evaluates social states by means of the value $\frac{f(n)}{n}\sum_{i=1}^{n}[g(u_i) - g(\alpha)]$, where $\alpha \in \mathbb{R}_+$ and f is a non-decreasing positive-valued function of population size. Rank-discounted utilitarianism proposed by Asheim and Zuber (2014) cannot escape the dilemma either. It employs the value $\sum_{r=1}^{n} \beta^r [g(u_{[r]}) - g(\alpha)]$, where $\beta \in (0, 1)$ and $u_{[\,]}$ is a non-decreasing rearrangement of u; see also Asheim and Zuber (forthcoming) and Pivato (2020).

Avoidance of the repugnant conclusion is a relatively weak requirement because it is based on existential quantifiers. However, the proof of part (a) of Theorem 4 reveals that our orderings satisfy specific stronger variants. For instance, there exists $\xi \in \mathbb{R}_{++}$ such that, for all $n \in \mathbb{N}$, for all $\varepsilon \in (0, \xi)$, and for all $m \in \mathbb{N}$ with $m > n$,

$$(\underbrace{\xi, \ldots, \xi}_{n}) R (\underbrace{\varepsilon, \ldots, \varepsilon}_{m}).$$

This suggest that some existential quantifiers can be replaced with universal quantifiers.

There are alternative population principles that are capable of avoiding both the repugnant and sadistic conclusions. Prominent examples are given by maximin and leximin principles; see, for instance, Bossert (1990), Blackorby, Bossert, and Donaldson (1996, 2005), and Zuber (2018). Our observation that both the repugnant and the sadistic conclusion can be avoided by generalized critical-level sufficientarian orderings illustrates that this reconciliation can be achieved without having to give priority to the worst-off.

6 Conclusion

This paper provides an axiomatic analysis of sufficientarianism. We examine the relationship between critical levels and thresholds for sufficiency that result from the strong Pareto principle. Moreover, we characterize a class of sufficientarian orderings with the property that the critical level need not be equal to the threshold of sufficiency. At least one issue remains to be addressed—namely, the possibility of several thresholds, a notion that is explored in some of the recent literature on sufficientarianism; see, for instance, Casal (2007) and Huseby (2010). In these approaches, the lower (or lowest) threshold is assumed to represent the basic needs for a decent human life, while higher thresholds are intended to identify levels of a flourishing life. Integrating multiple thresholds into generalized critical-level sufficientarian orderings is a challenge to be addressed in future work. For this purpose, our new axiom of expansion independence may turn out to be helpful in extending the single-threshold method to multiple-threshold versions. A promising approach consists of combining the axiom with a plausible extension of absolute priority that respects multiple layers of thresholds.

We adopt a welfarist framework throughout this paper. In contrast, contributions such as those of Anderson (1999, 2007) apply sufficientarian ideas to the capability approach, according to which absolute priority is to be given to improvements in the capabilities of those who do not have enough. Each person's capability is given as a set in the space of functionings and, thus, capability distributions are to be compared. See also Alcantud, Mariotti, and Veneziani (2019) who examine sufficientarianism in the context of opportunities or chances of success. Clearly, these are perfectly legitimate alternatives but we do not pursue them. It seems to us that welfarism rests on a solid normative foundation, which is why we focus on the variant of sufficientarianism explored here.

Appendix: Independence of the axioms

This appendix establishes the independence of the axioms used in Theorem 2. Let $g\colon \mathbb{R} \to \mathbb{R}$ be an increasing and strictly concave (and, thus, continuous) function and define the ordering R_1 as follows. For all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uR_1v \;\Leftrightarrow\; \sum_{i=1}^{n}[g(u_i) - g(\theta)] \geq \sum_{i=1}^{m}[g(v_i) - g(\theta)],$$

that is, R_1 is the critical-level generalized-utilitarian ordering associated with g and the critical level of utility given by θ. The ordering R_1 satisfies all of the axioms in the theorem except absolute priority.

Let $\alpha_1 = \theta + 1$ and $\alpha_i = \theta$ for all $i \in \mathbb{N} \setminus \{1\}$. Let $g \in G$ and define, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$, uR_2v if and only if

$$\sum_{i\in L^n(u)}[g(u_i) - g(\theta)] > \sum_{i\in L^m(v)}[g(v_i) - g(\theta)]$$

or

$$\textstyle\sum_{i\in L^n(u)}[g(u_i) - g(\theta)] = \sum_{i\in L^m(v)}[g(v_i) - g(\theta)] \text{ and}$$
$$\textstyle\sum_{i\in H^n(u)\cup E^n(u)}[g(u_i) - g(\alpha_i)] \geq \sum_{i\in H^m(v)\cup E^m(v)}[g(v_i) - g(\alpha_i)].$$

The ordering R_2 satisfies all of our axioms except anonymity. Consider $u = (\theta + 1, \theta - 1)$ and $v = (\theta - 1, \theta + 1)$. Then, vP_2u follows since $g(u_2) = g(v_1)$ and $g(\theta + 1) - g(\alpha_1) < g(\theta + 1) - g(\alpha_2)$.

Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous and increasing function that is strictly concave on $\mathbb{R}_L$ and convex on $\mathbb{R}_{HE}$. Define the ordering R_3 as follows. For all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$, uR_3v if and only if

$$\sum_{i\in L^n(u)}[g(u_i) - g(\theta)] > \sum_{i\in L^m(v)}[g(v_i) - g(\theta)]$$

or

$$\sum_{i\in L^n(u)}[g(u_i) - g(\theta)] = \sum_{i\in L^m(v)}[g(v_i) - g(\theta)] \text{ and } \sum_{i\in H^n(u)}[g(u_i) - g(\theta)] \leq \sum_{i\in H^m(v)}[g(v_i) - g(\theta)].$$

The ordering R_3 satisfies all axioms except strong Pareto.

Let $g \in G$ and $\delta_L \geq \sup_{a \in \mathbb{R}_L} g(a)$, and define, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$, uR_4v if and only if

$$\sum_{i \in L^n(u)} [g(u_i) - \delta_L] > \sum_{i \in L^m(v)} [g(v_i) - \delta_L]$$

or

$$\sum_{i \in L^n(u)} [g(u_i) - \delta_L] = \sum_{i \in L^m(v)} [g(v_i) - \delta_L] \text{ and}$$
$$\frac{1}{|H^n(u) \cup E^n(u)|} \sum_{i \in H^n(u) \cup E^n(u)} g(u_i) \geq \frac{1}{|H^m(v) \cup E^m(v)|} \sum_{i \in H^m(v) \cup E^m(v)} g(v_i).$$

The ordering R_4 satisfies all of our axioms except existence independence.

To present an example that shows that continuity below the threshold is not implied, we need some additional notation and definitions. We use $\mathbf{1}_n$ to denote the vector that consists of $n \in \mathbb{N}$ ones. For $n \in \mathbb{N}$ and $u \in \mathbb{R}^n$, let $(u_{(1)}, \ldots, u_{(n)})$ denote a permutation of u such that $u_{(1)} \geq \ldots \geq u_{(n)}$; that is, the utilities in such a permutation are ranked from highest to lowest, with ties being broken arbitrarily. For each $n \in \mathbb{N}$, the leximin ordering R^n_{lex} on $\mathbb{R}^n$ is defined by letting, for all $u, v \in \mathbb{R}^n$, $uR^n_{lex}v$ if and only if u is a permutation of v or there exists $j \in \{1, \ldots, n\}$ such that $u_{(i)} = v_{(i)}$ for all $i > j$ and $u_{(j)} > v_{(j)}$. Given a threshold level θ, the corresponding critical-level leximin ordering R_{lex} on Ω is defined by letting, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$\begin{aligned} uR_{lex}v \quad \Leftrightarrow \quad & [n = m \text{ and } uR^n_{lex}v] \text{ or} \\ & [n > m \text{ and } uR^n_{lex}(v, \theta\mathbf{1}_{n-m})] \text{ or} \\ & [n < m \text{ and } (u, \theta\mathbf{1}_{m-n})R^m_{lex}v]. \end{aligned}$$

For all $n \in \mathbb{N}$ and for all $u \in \mathbb{R}^n$, define

$$K^n(u) = \{i \in \{1, \ldots, n\} \mid u_i < \theta - 1\}.$$

For all $n \in \mathbb{N}$ and for all $u \in \mathbb{R}^n$, if $K^n(u) = \varnothing$, then we write $\omega_\varnothing = (u_i)_{i \in K^n(u)}$. Using the leximin orderings R^n_{lex} for n-dimensional utility distributions and the critical-level leximin ordering R_{lex} associated with the threshold θ, we define the extended critical-level leximin ordering R^*_{lex} on $\cup_{n \in \mathbb{N}}(-\infty, \theta - 1)^n \cup \{\omega_\varnothing\}$ as follows. For all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$\begin{aligned} (u_i)_{i \in K^n(u)} R^*_{lex} (v_i)_{i \in K^m(v)} \Leftrightarrow & [K^n(u) = K^m(v) = \varnothing] \text{ or} \\ & [K^n(u) = \varnothing \neq K^m(v) \text{ and } (\theta\mathbf{1}_{|K^m(v)|}) R^{|K^m(v)|}_{lex} (v_i)_{\in K^m(v)}] \text{ or} \\ & [K^n(u) \neq \varnothing = K^m(v) \text{ and } (u_i)_{i \in K^n(u)} R^{|K^n(u)|}_{lex} (\theta\mathbf{1}_{|K^n(u)|})] \text{ or} \\ & [K^n(u) \neq \varnothing \neq K^m(v) \text{ and } (u_i)_{i \in K^n(u)} R_{lex} (v_i)_{i \in K^m(v)}]. \end{aligned}$$

Let $g\colon \mathbb{R} \to \mathbb{R}$ be increasing and strictly concave (and, thus, continuous). We define the ordering R_5 as follows. For all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$, uR_5v if and only if

$$(u_i)_{i\in K^n(u)} P^*_{lex} (v_i)_{i\in K^m(v)}$$

or

$$(u_i)_{i\in K^n(u)} I^*_{lex} (v_i)_{i\in K^m(v)} \text{ and } \sum_{i\in L^n(u)\setminus K^n(u)} [g(u_i) - g(\theta)] > \sum_{i\in L^m(v)\setminus K^m(v)} [g(v_i) - g(\theta)]$$

or

$$(u_i)_{i\in K^n(u)} I^*_{lex} (v_i)_{i\in K^m(v)} \text{ and } \sum_{i\in L^n(u)\setminus K^n(u)} [g(u_i) - g(\theta)] = \sum_{i\in L^m(v)\setminus K^m(v)} [g(v_i) - g(\theta)] \text{ and } \sum_{i\in H^n(u)} [u_i - \theta] \geq \sum_{i\in H^m(v)} [v_i - \theta].$$

The ordering R_5 satisfies all the axioms except continuity below the threshold. Showing that expansion independence is satisfied is not trivial and, thus, we provide a detailed proof. Let $n, m \in \mathbb{N}$. We show that there exist $k \in \mathbb{N}$ with $k > \max\{n, m\}$ and $\lambda, \mu \in [\theta - 1, \theta) \subset \mathbb{R}_L$ that satisfy the requisite condition. Note that any $k \in \mathbb{N}$ with $k > \max\{n, m\}$ and any $\lambda, \mu \in [\theta - 1, \theta)$ satisfy that, for all $u \in \mathbb{R}^n_L$ and for all $v \in \mathbb{R}^m_L$,

$$(u_i)_{i\in K^n(u)} = (\bar{u}_i)_{i\in K^k(\bar{u})} \text{ and } (v_i)_{i\in K^m(v)} = (\bar{v}_i)_{i\in K^k(\bar{v})} \tag{15}$$

where $\bar{u} = (u, \underbrace{\lambda, \ldots, \lambda}_{k-n})$ and $\bar{v} = (v, \underbrace{\mu, \ldots, \mu}_{k-m})$. Thus, by the definition of R_5, it suffices to show that there exist $k \in \mathbb{N}$ with $k > \max\{n, m\}$ and $\lambda, \mu \in [\theta - 1, \theta)$ such that, for all $u \in \mathbb{R}^n_L$ and for all $v \in \mathbb{R}^m_L$,

$$(u_i)_{i\in K^n(u)} I^*_{lex} (v_i)_{i\in K^m(v)} \Leftrightarrow (\bar{u}_i)_{i\in K^k(\bar{u})} I^*_{lex} (\bar{v}_i)_{i\in K^k(\bar{v})} \tag{16}$$

and

$$\begin{aligned} &\sum_{i\in L^n(u)\setminus K^n(u)} [g(u_i) - g(\theta)] \geq \sum_{i\in L^m(v)\setminus K^m(v)} [g(v_i) - g(\theta)] \\ &\qquad \Leftrightarrow \sum_{i\in L^k(\bar{u})\setminus K^k(\bar{u})} [g(\bar{u}_i) - g(\theta)] \geq \sum_{i\in L^k(\bar{v})\setminus K^k(\bar{v})} [g(\bar{v}_i) - g(\theta)] \end{aligned} \tag{17}$$

where $\bar{u} = (u, \underbrace{\lambda, \ldots, \lambda}_{k-n})$ and $\bar{v} = (v, \underbrace{\mu, \ldots, \mu}_{k-m})$.

First, note that, for all $u \in \mathbb{R}^n_L$ and for all $v \in \mathbb{R}^m_L$,

$$\begin{aligned} (u_i)_{i\in K^n(u)} I^*_{lex} (v_i)_{i\in K^m(v)} \Leftrightarrow\ & [K^n(u) = K^m(v) = \varnothing] \text{ or} \\ & [(u_i)_{i\in K^n(u)} \text{ is a permutation of } (v_i)_{i\in K^m(v)}]. \end{aligned}$$

If $(u_i)_{i\in K^n(u)}$ is a permutation of $(v_i)_{i\in K^m(v)}$, then $|K^n(u)| = |K^m(v)|$. Thus, if $n = m$, conditions (16) and (17) are satisfied for any $k > \max\{n, m\}$ and $\lambda, \mu \in [\theta - 1, \theta)$ such that $\lambda = \mu$.

Next, we consider the case where $n \neq m$. Without loss of generality, suppose that $m > n$. We begin by showing that there exist $k \in \mathbb{N}$ with $k > m$ and $\lambda, \mu \in [\theta-1, \theta)$ such that

$$g(\theta) = \frac{(k-n)g(\lambda) - (k-m)g(\mu)}{m-n}. \tag{18}$$

Since g is increasing on $\mathbb{R}$, $g(a) < g(\theta)$ for all $a \in \mathbb{R}_L$. Furthermore, g is not bounded below because of strict concavity. Thus, there exist (sufficiently large) $k \in \mathbb{N}$ with $k > m$ and $\lambda \in (\theta - 1, \theta)$ such that

$$g(\theta - 1) < g(\lambda) - \frac{m-n}{k-m}[g(\theta) - g(\lambda)] < g(\lambda) < g(\theta).$$

Since g is continuous on $\mathbb{R}$, it follows from the intermediate value theorem that there exists $\mu \in (\theta - 1, \lambda) \subset [\theta - 1, \theta)$ such that $g(\mu) = g(\lambda) - \frac{m-n}{k-m}[g(\theta) - g(\lambda)]$, that is,

$$g(\theta) = \frac{(k-n)g(\lambda) - (k-m)g(\mu)}{m-n}.$$

We now show that, for k, λ, and μ that satisfy (18), conditions (16) and (17) are satisfied. By (15), it is straightforward that (16) is satisfied. To show that (17) holds, let $u \in \mathbb{R}^n_L$ and $v \in \mathbb{R}^m_L$. Since $(u_i)_{i\in K^n(u)} I^*_{lex} (v_i)_{i\in K^m(v)}$ implies $|K^m(v)| = |K^n(u)|$, we can assume $m - n = |L^m(v) \setminus K^m(v)| - |L^n(u) \setminus K^n(u)|$. Let $\bar{u} = (u, \underbrace{\lambda, \ldots, \lambda}_{k-n})$ and $\bar{v} = (u, \underbrace{\mu, \ldots, \mu}_{k-m})$. Then, we obtain

$$\begin{aligned}
&\sum_{i\in L^n(u)\setminus K^n(u)} [g(u_i) - g(\theta)] \geq \sum_{i\in L^m(v)\setminus K^m(v)} [g(v_i) - g(\theta)] \\
&\quad\Leftrightarrow \sum_{i\in L^n(u)\setminus K^n(u)} g(u_i) + (m-n)g(\theta) \geq \sum_{i\in L^m(v)\setminus K^m(v)} g(v_i) \\
&\quad\Leftrightarrow \sum_{i\in L^k(\bar{u})\setminus K^k(\bar{u})} g(\bar{u}_i) \geq \sum_{i\in L^k(\bar{v})\setminus K^k(\bar{v})} g(\bar{v}_i) \\
&\quad\Leftrightarrow \sum_{i\in L^k(\bar{u})\setminus K^k(\bar{u})} [g(\bar{u}_i) - g(\theta)] \geq \sum_{i\in L^k(\bar{v})\setminus K^k(\bar{v})} [g(\bar{v}_i) - g(\theta)],
\end{aligned}$$

since $|L^k(\bar{u}) \setminus K^k(\bar{u})| = |L^k(\bar{v}) \setminus K^k(\bar{v})|$. Thus, R_5 satisfies expansion independence.

Let $g \in G$ and $\delta_L \geq \sup_{a\in\mathbb{R}_L} g(a)$, and define the ordering R_6 as follows. For all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uR_6v \;\Leftrightarrow\; \sum_{i\in L^n(u)} [g(u_i) - \delta_L] > \sum_{i\in L^m(v)} [g(v_i) - \delta_L]$$

or

$$\sum_{i\in L^n(u)} [g(u_i)-\delta_L] = \sum_{i\in L^m(v)} [g(v_i)-\delta_L] \text{ and } (u_i)_{i\in H^n(u)\cup E^n(u)} \, R_{lex} \, (v_i)_{i\in H^m(v)\cup E^m(v)},$$

where R_{lex} is the critical-level leximin ordering associated with θ. The ordering R_6 satisfies all of the axioms except continuity above and at the threshold.

Let $g \in G$ and $\delta_L \geq \sup_{a\in\mathbb{R}_L} g(a)$. Define, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$, uR_7v if and only if

$$\sum_{i\in L^n(u)} [g(u_i)-\delta_L] > \sum_{i\in L^m(v)} [g(v_i)-\delta_L]$$

or

$$\sum_{i\in L^n(u)} [g(u_i)-\delta_L)] = \sum_{i\in L^m(v)} [g(v_i)-\delta_L] \text{ and } |H^n(u)\cup E^n(u)| > |H^m(v)\cup E^m(v)|$$

or

$$\sum_{i\in L^n(u)} [g(u_i)-\delta_L)] = \sum_{i\in L^m(v)} [g(v_i)-\delta_L] \text{ and } |H^n(u)\cup E^n(u)| = |H^m(v)\cup E^m(v)|$$
$$\text{and } \sum_{i\in H^n(u)\cup E^n(u)} u_i \geq \sum_{i\in H^m(v)\cup E^m(v)} v_i.$$

The ordering R_7 does not satisfy weak existence of critical levels. All other axioms are satisfied.

Let $g \in G$ and define, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$, uR_8v if and only if

$$|L^n(u)| < |L^m(v)|$$

or

$$|L^n(u)| = |L^m(v)| \text{ and } \sum_{i\in L^n(u)} g(u_i) > \sum_{i\in L^m(v)} g(v_i)$$

or

$$|L^n(u)| = |L^m(v)| \text{ and } \sum_{i\in L^n(u)} g(u_i) = \sum_{i\in L^m(v)} g(v_i)$$
$$\text{and } \sum_{i\in H^n(u)} [g(u_i)-g(\theta)] \geq \sum_{i\in H^m(v)} [g(v_i)-g(\theta)].$$

The ordering R_8 violates expansion independence. This can be verified as follows. Let $n = 1$ and $m = 2$. Expansion independence asserts that there exist $k \in \mathbb{N}\setminus\{1, 2\}$ and $\lambda, \mu \in \mathbb{R}_L$ such that, for all $u \in \mathbb{R}$ and for all $v \in \mathbb{R}^2$, uRv if and only if

$(u, \lambda \mathbf{1}_{k-1}) R (v, \mu \mathbf{1}_{k-2})$. Let $u \in \mathbb{R}_L$ and $v = (\lambda, \lambda) \in \mathbb{R}_L^2$. By the definition of R_8, $u P_8 v$ follows. On the other hand, $(v, \mu \mathbf{1}_{k-2}) = (\lambda, \lambda, \mu \mathbf{1}_{k-2}) P_7 (u, \lambda \mathbf{1}_{k-1})$ follows if

$$g(u) + (k-1)g(\lambda) < 2g(\lambda) + (k-2)g(\mu) \Leftrightarrow g(u) < g(\lambda) - (k-2)[g(\lambda) - g(\mu)].$$

Note that $\lim_{a \to -\infty} g(a) = -\infty$ since $g \in G$. Thus, there exists $u \in \mathbb{R}_L$ that satisfies this inequality. This means that R_8 violates expansion independence. All other axioms are satisfied.

Define, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$, $u R_9 v$ if and only if

$$\sum_{i \in L^n(u)} [u_i - \theta] > \sum_{i \in L^m(v)} [v_i - \theta]$$

or

$$\sum_{i \in L^n(u)} [u_i - \theta] = \sum_{i \in L^m(v)} [v_i - \theta] \text{ and } \sum_{i \in H^n(u)} [u_i - \theta] \geq \sum_{i \in H^m(v)} [v_i - \theta].$$

The ordering R_9 does not satisfy the principle of transfers below the threshold. All other axioms are satisfied.

Finally, let $g \colon \mathbb{R} \to \mathbb{R}$ be a continuous and increasing function that is strictly concave on $\mathbb{R}_L$ and strictly convex on $\mathbb{R}_{HE}$. Define, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$, $u R_{10} v$ if and only if

$$\sum_{i \in L^n(u)} [g(u_i) - g(\theta)] > \sum_{i \in L^m(v)} [g(v_i) - g(\theta)]$$

or

$$\sum_{i \in L^n(u)} [g(u_i) - g(\theta)] = \sum_{i \in L^m(v)} [g(v_i) - g(\theta)] \text{ and } \sum_{i \in H^n(u)} [g(u_i) - g(\theta)] \geq \sum_{i \in H^m(v)} [g(v_i) - g(\theta)].$$

The ordering R_{10} does not satisfy the weak principle of transfers above and at the threshold. All other axioms are satisfied.

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