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Abstract. This paper provides a characterization of a new class of ordinal poverty measures that are defined by means of the aggregate generalized poverty gap. To be precise, we propose to use the sum of the differences between the transformed fixed poverty line and the transformed level of income of each person below the line as our measure. If the transformation is strictly concave, the resulting measure is strictly inequality averse with respect to the incomes of the poor. In analogy to some existing results on inequality measurement, we show that the only relative (scale-invariant) members of our class are based on specific means of order r. We suggest that our poverty measures can be interpreted as inverse welfare indicators for the poor and derive relative and absolute ethical indices of inequality that correspond to them. In an empirical analysis, we compare the logarithmic variant of our index to two well-established alternative orderings. Unlike numerous indices that appear in the earlier literature, ours do not explicitly depend on the number of poor or on the total population size, thereby ruling out any direct influence of the head-count ratio on poverty comparisons. *Journal of Economic Literature* Classification Nos.: D31, D63.

1 Introduction

The axiomatic approach to the measurement of poverty is founded on Sen's (1976) seminal contribution that paved the way for numerous subsequent studies, many of which are summarized in Zheng's (1997, 2000) comprehensive surveys. A collection of more recent essays on the topic is provided in the volume edited by Betti and Lemmi (2014).

One of the cornerstones of poverty measurement is the well-established focus axiom (Sen, 1976). Basically, this property requires that poverty does not depend on the incomes of the non-poor—that is, the incomes of those above the poverty line. Focus plays an important role in our approach as well, and we broaden its scope to address some additional issues that may emerge. To illustrate, suppose that an individual above the poverty line in a society moves to another society where (s)he is also among the non-poor. All else unchanged, the effect of this move on the recipient country is that overall population size increases and the incidence of poverty remains unchanged. According to most existing poverty measures, poverty in the recipient country decreases since they measure the seriousness of poverty on a per-capita basis. Moreover, if the measure in question possesses some mild continuity property, the level of poverty can decrease even if the move is accompanied by a regressive transfer from a poor person to the rich migrant (through his or her economic activities, for example). By the same token, poverty becomes higher in the country from which this rich person moved away. That is, the majority of the standard poverty measures are sensitive to the existence of rich people in a society. It is natural that the migration of poor people affects local poverty levels. However, is the sensitivity to the existence of non-poor individual as illustrated above plausible? If the answer is in the affirmative, a possible approach to poverty reduction is to increase the number of rich people while leaving the situation of the poor unchanged. We consider this an ethically unattractive position and, therefore, propose an alternative approach that does not exhibit this feature.

Even if one is primarily concerned with global poverty comparisons, the above illustration translates easily into a setting in which migration is replaced by someone above the poverty line being brought into existence. Again, the question is whether such a population change decreases poverty even though none of the poor is affected by this change. To be clear, we agree that adding a person with a high level of well-being or income to an otherwise unchanged population is desirable from an overall welfare perspective. But, as most—if not all—contributors in this area would readily agree, an increase in social welfare is not identical to a decrease in poverty. Therefore, our objection to the dependence of poverty on the existence of non-poor people does not in any way conflict with the postulate that there may be a positive welfare effect associated with increasing the number of those who are well-off in a society.

Insensitivity with respect to the existence of the non-poor is closely associated with the spirit of Sen's (1976) focus axiom. Sen's original axiom requires that a change in the income level of a rich person does not affect the poverty level as long as the rich individual in question remains above the poverty line. This type of insensitivity is satisfied by almost all poverty measures. We impose an extended version of the focus axiom, which requires that the existence of people above the poverty line does not affect the poverty level. The existence-insensitivity variant is stronger than the income-insensitivity version represented by the focus axiom. We show that our extended focus axiom leads to what we call the aggregate generalized poverty gap when combined with some normatively plausible axioms for a poverty ranking. The aggregate generalized poverty gap uses the sum of the differences between the transformed fixed poverty line and the transformed level of income of each person below the line as the criterion to assess poverty. The use of such a poverty ranking yields a quite different picture from that provided by traditional measures that do not possess the existence-independence feature of our extended focus axiom. This is illustrated in Section 6 by means of an application to the trend in poverty in the United States between 1991 and 2017.

We stress that, although our main interest lies in the analysis of poverty as a global phenomenon, nothing prevents us from applying our measures to specific countries (or, more generally, to specific geographic regions). As a consequence of their additive structure, the members of our class are decomposable so that inter-regional comparisons can be performed without any concerns regarding possible inconsistencies. See Foster, Greer, and Thorbecke (1984) for a general notion of decomposability in the context of poverty measurement.

We assume the poverty line to be fixed in the sense that it does not depend on the income distribution under consideration. This assumption is widely employed in the theoretical literature on poverty measurement. Moreover, it is consistent with the practice of determining the poverty line based on basic needs; for instance, the official poverty line of the United States, that of Canada since 2018, and the International Poverty Line set by the World Bank are determined according to this convention. We are well aware that the existing literature on the subject, in particular, the literature on empirical assessment of poverty, contains contributions that employ endogenous poverty lines; perhaps the most prominent examples are those in which the poverty line is determined as a proportion of an average income (such as the median) within a country—or within a group of countries. Approaches of this nature help draw an insight into the problem of social exclusion. In our view, however, those approaches suffer from potentially quite serious shortcomings, especially if the poverty line is determined as a percentage of an aggregate income measure in a group of countries. This method can have quite counterintuitive effects. In particular, there is a distinct possibility that the relative poverty ranking of two countries depends on whether another country is a member of the group—a consideration that can be quite relevant indeed, for example in the context of Brexit. Ravallion and Chen (2011, 2019) perform a global comparison of poverty by introducing an endogenous poverty line; they use a relative-income approach to arrive at a poverty line that is increasing in the average income in each country. However, Ravallion and Chen (2011, 2019) assume that a poverty line is globally fixed in the utility dimension, although the line is endogenous in the income dimension. Thus, their approach is compatible with our framework if a utility function is explicitly incorporated.

The poverty orderings that we advocate are motived by Blackorby and Donaldson's (1984) critical-level generalized-utilitarian population principles; see also Blackorby, Bossert, and Donaldson (2005). Critical-level utilitarianism uses as a social criterion the sum of the individual differences between lifetime well-being (utility) and a fixed critical level that represents a minimally acceptable standard of living. The interpretation of the critical level is straightforward: the ceteris-paribus addition of a person to an existing population is socially desirable if and only if the lifetime utility of the new individual is above the critical level. The generalized counterpart of critical-level utilitarianism employs transformed utilities, thereby allowing for the possibility of incorporating inequality aversion. This principle has also been applied in the context of sufficientarian theories of justice. A sufficientarian criterion pays special attention to those below a given threshold level of well-being but may allow the well-being of those who are at or above the threshold to matter to a lesser extent. In contrast, a poverty measure focuses exclusively on those who are below the poverty line. Sufficientarianism was first explored by Frankfurt (1987) who proposed to use the (relative) number of those below the threshold to define a sufficientarian ranking of income or utility distributions. This approach has been criticized in numerous subsequent contributions, including those of Crisp (2003), Brown (2005), Casal (2007), Huseby (2012), and Hirose (2016), to name but a few. Once the critical level is reinterpreted as the threshold, it appears quite natural to define a sufficientarian criterion as a principle that is based on the aggregate (transformed) shortfalls from the threshold in the first instance and use the aggregate surplus of those above the threshold as a tie-breaking device. This approach is explored by Bossert, Cato, and Kamaga (2020a,b).

Returning to the context of poverty measurement, the notion of aggregate generalized gaps appears again as a natural candidate to assess the situation of the poor. Unlike a sufficientarian principle, however, poverty measurement restricts exclusive attention to the levels of well-being or income of those below the poverty line—there is no reason to consult the part of the distribution that applies to the non-poor. This viewpoint is unambiguously reflected in Sen's (1976) focus axiom which we, along with the majority of the subsequent literature, fully endorse (and even strengthen, as hinted at earlier). To be specific, the poverty orderings we propose and characterize are based on the aggregate difference between the transformed poverty line and the transformed incomes of the poor. An important subclass is obtained if the strict Pigou-Dalton transfer principle (Pigou, 1912, and Dalton, 1920) for the poor is added as a requirement. In this case, the transformation applied to the incomes and to the poverty line must be strictly concave.

It has been understood for some time now that coarse measures such as the head-count ratio are inadequate to truly capture the severity and the hardships still suffered by too many on this planet and, therefore, it is difficult to find advocates of the exclusive use of this ratio as an indicator of poverty. However, the head-count ratio continues to play a role in combination with other attributes of an income distribution, such as various notions of the aggregate income gap of the poor. A novel feature of this paper is that we deviate from this practice and propose a class of poverty measures that do not directly depend on the number of poor or the total population size. Of course, our measures still satisfy the standard monotonicity requirements usually imposed on an index of poverty. In particular, if there is a ceteris-paribus increase in the number of poor, this will lead to an increase in the value of our poverty measures because the aggregate generalized poverty gap increases as a consequence of such a change. The only impact that is absent in the case of our measures is an additional increase that results directly from the increase in the number of those below the poverty line—or from the associated increase in the head-count ratio. But this seems to be a perfectly acceptable attribute of an index: as long as the requisite monotonicity properties are respected (which certainly is true in this case), there is nothing wrong with the absence of an additional (direct) effect through the head-count ratio.

Because our measures do not depend on the number of poor per se, they can be interpreted as reverse indicators of well-being for the poor. This allows for a plausible and straightforward link to be established between our indicators and ethical measures of inequality for those below the poverty line. Ours is certainly not the first contribution that examines the normative connections between poverty and inequality (most notably, Blackorby and Donaldson, 1980b, and Vaughn, 1987, are precursors in that regard) but the actual relationship between poverty and inequality is different in our setting as a consequence of the observation that our measures do not depend on attributes such as the head-count ratio.

Our basic definitions are presented in Section 2, and Section 3 introduces the axioms that are used in our characterizations. The theoretical results of the paper appear in Section 4. In particular, we provide a characterization of our new class and the subclasses that result if we add strict inequality aversion and scale invariance to our list of axioms. Section 5 explores possible links between our class of measures and inequality among the poor, and Section 6 provides an application by offering a comparison between some existing poverty measures and the logarithmic ordering that belongs to our class. Section 7 concludes.

2 Preliminaries

Let $\mathbb{N}$ be the set of all positive integers. The set of all positive real numbers is denoted by $\mathbb{R}_{++}$. For $n \in \mathbb{N}$, we use $\mathbf{1}^n$ to denote the vector that consists of n ones. We adopt the notational convention that the sum taken over an empty set is equal to zero. An income distribution for a population of $n \in \mathbb{N}$ individuals is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$. In order to allow for poverty comparisons across different time periods (and across different geographical regions if so desired), we permit population size to vary and, thus, we consider the set of income distributions given by the union $\Omega = \cup_{n \in \mathbb{N}} \mathbb{R}^n_{++}$. Although many of our observations can be extended to non-negative incomes, zero incomes cannot be accommodated for an important characterization result that involves relative poverty measures. For this reason, we assume incomes to be positive throughout.

To identify the set of those who are poor, we employ a fixed level of income that we use as the poverty line $z \in \mathbb{R}_{++}$. For the class of poverty orderings characterized here, it does not matter whether we use the weak definition of the poor or the strong definition of the poor. According to the weak definition, everyone below the poverty line is considered poor but those whose incomes are equal to the poverty line are not; in contrast, the strong definition of the poor counts those located at the poverty line among the poor as well. As discussed by Donaldson and Weymark (1986), the strong definition of the poor may lead to impossibility results when it comes to the compatibility of various sets of properties in a fixed-population setting. We use the strong definition of the poor in our formal definitions for convenience but reiterate that the weak definition is equivalent in the context of our axioms and the orderings they characterize and, thus, the issues raised by Donaldson and Weymark (1986) are not a serious cause for concern in our case. For a poverty line $z \in \mathbb{R}_{++}$, the set of the poor in a distribution $x = (x_1, \ldots, x_n) \in \Omega$ is given by

$$\mathcal{P}(z; x) = \{i \in \{1, \ldots, n\} \mid x_i \leq z\}.$$

Given a poverty line and an income distribution such that the set of the poor is non-empty, we use $x^p = (x_i)_{i \in \mathcal{P}(z;x)}$ to denote the subvector of $x \in \Omega$ that corresponds to the incomes of the poor for the poverty line $z \in \mathbb{R}_{++}$. Letting x^{np} denote the subvector of $x \in \Omega$ that corresponds to the non-poor, a distribution $x \in \Omega$ can be partitioned into these two constituent parts so that $x = (x^p, x^{np})$.

Although we work with poverty lines that do not depend on the distribution under consideration, it is important to allow this line to vary. This is necessary in order to formulate properties such as the well-established requirement that an indicator of poverty be relative. Thus, we define a poverty ordering R to be a transitive and complete binary relation on the set of pairs $((z; x), (z; y))$ with $z \in \mathbb{R}_{++}$ and $x, y \in \Omega$. For a poverty line $z \in \mathbb{R}_{++}$ and two income distributions $x, y \in \Omega$, we interpret $(z; x)R(z; y)$ to mean that the poverty level of x is higher than or equal to that of y, given the poverty line z. The asymmetric and symmetric parts of R are denoted by P and I, that is, for a poverty line z, we write $(z; x)P(z; y)$ to say that there is more poverty in x than in y, and $(z; x)I(z; y)$ indicates that the poverty levels for the distributions x and y are equal.

A poverty ordering R is a generalized poverty-gap ordering if there exists a continuous and increasing function $g \colon \mathbb{R}_{++} \to \mathbb{R}$ such that, for all $z \in \mathbb{R}_{++}$ and for all $x, y \in \Omega$,

$$(z; x)R(z; y) \;\Leftrightarrow \sum_{i \in \mathcal{P}(z;x)} (g(z) - g(x_i)) \geq \sum_{i \in \mathcal{P}(z;y)} (g(z) - g(y_i)) \, .$$

If the weak definition of the poor is employed (that is, if those whose incomes are equal to the poverty line are not considered poor), the generalized poverty-gap criterion remains unchanged because the contributions to overall poverty of these individuals is equal to $g(z) - g(z) = 0$. This equivalence of the weak and strong definitions of the poor attributable to the fact that our orderings do not directly depend on the number of the poor or on total population size. As a consequence, the problems traditionally associated with the strong definition (see Donaldson and Weymark, 1986) are not of any concern in our setting.

Recall that, according to our convention, the sum $\sum_{i \in \mathcal{P}(z;x)} (g(z) - g(x_i))$ assumes the value zero if $\mathcal{P}(z; x) = \varnothing$. Thus, if there are no poor in the distribution x for the poverty line z, poverty is as low as possible because the value of this sum is positive whenever the set of poor is non-empty.

The generalized poverty-gap orderings are inspired by the critical-level generalized-utilitarian population principles introduced by Blackorby and Donaldson (1984); see also Blackorby, Bossert, and Donaldson (2005) for a comprehensive treatment.

3 Axioms

As is standard in the literature on poverty measurement (and on social index numbers in general), we assume that the identities of the income recipients are ethically irrelevant. The axiom of anonymity formalizes this assumption.

Anonymity. For all $z \in \mathbb{R}_{++}$ and for all $x, y \in \Omega$, if y is obtained from x by applying a permutation to the components of x, then $(z; x)I(z; y)$.

The focus axiom is a standard property in fixed-population poverty measurement; see Sen (1976). It requires that the incomes of the non-poor do not influence the poverty ranking. We use a strengthening that applies in a variable-population setting. This strengthening, which implies that not only the incomes of the non-poor but also their existence are irrelevant for poverty assessments, is a key property for our argument.

Extended Focus. For all $z \in \mathbb{R}_{++}$, for all $x \in \Omega$, for all $n, m \in \mathbb{N}$, for all $y \in \mathbb{R}^n_{++}$, and for all $w \in \mathbb{R}^m_{++}$, if $y_i \geq z$ for all $i \in \{1, \ldots, n\}$ and $w_i \geq z$ for all $i \in \{1, \ldots, m\}$, then

$$(z; x)I(z; (x, y)) \text{ and } (z; (x, y))I(z; (x, w)).$$

We note that, by restricting y, w to be of the same population size, this axiom implies the original focus axiom, which requires that $(z; (x, y))I(z; (x, w))$ for all $z \in \mathbb{R}_{++}$, for all $x \in \Omega$, for all $n \in \mathbb{N}$, and for all $y, w \in \mathbb{R}^n_{++}$ such that $y_i \geq z$ and $w_i \geq z$ for all $i \in \{1, \ldots, n\}$. Consider the special case of the extended focus axiom that is obtained for $n = 1$ and $y = (z)$. If a given population is augmented by an individual whose income is equal to the poverty line, the resulting distribution is associated with the same level of poverty as the original. This feature is related to the observation that the generalized poverty-gap orderings can equivalently be expressed in terms of the weak definition of the poor. As mentioned earlier, extended focus implies that not only the incomes of the non-poor but also their existence are irrelevant for the poverty ranking. This property is in stark contrast with the nonpoverty growth axiom proposed by Kundu and Smith (1983), which requires poverty to decrease if a non-poor person is brought into existence.

Another standard requirement demands poverty to be decreasing in the income levels of the poor.

Strict monotonicity below and at the poverty line. For all $z \in \mathbb{R}_{++}$, for all $n \in \mathbb{N}$, for all $x, y \in \mathbb{R}^n_{++}$, and for all $i \in \{1, \ldots, n\}$, if $z \geq x_i > y_i$ and $x_j = y_j$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$, then $(z; y)P(z; x)$.

The next axiom is a separability property that is familiar from the literature on population ethics. For our purposes, it is sufficient to restrict its scope to fixed population sizes but it can be extended to cover variable-population situations as well; see, for example, Blackorby and Donaldson (1984) or Blackorby, Bossert, and Donaldson (2005, Chapter 5).

Fixed-number independence. For all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x, y \in \mathbb{R}^n_{++}$, and for all $w, s \in \mathbb{R}^m_{++}$,

$$(z; (x, w))R(z; (y, w)) \Leftrightarrow (z; (x, s))R(z; (y, s)).$$

Continuity is a well-established robustness property. It ensures that 'small' changes in an income distribution do not lead to 'large' changes in the poverty ordering. We restrict its scope to incomes below or at the poverty line because, by the extended focus axiom, the changes in incomes of those at or above the poverty line are irrelevant.

Continuity below and at the poverty line. For all $z \in \mathbb{R}_{++}$, for all $n \in \mathbb{N}$, and for all $x \in (0, z]^n$, the sets $\{y \in (0, z]^n \mid (z; y)R(z; x)\}$ and $\{y \in (0, z]^n \mid (z; x)R(z; y)\}$ are closed in $(0, z]^n$.

We also impose an equity property that applies to the incomes of those below and at the poverty line. The strict transfer principle (see Pigou, 1912, and Dalton, 1920) restricted to the poor requires that a transfer from a poor person to one who is even poorer leads to a reduction in poverty, provided that the relative ranks of the two individuals are unchanged and no one else's income is changed.

Principle of transfers below and at the poverty line. For all $z \in \mathbb{R}_{++}$, for all $n \in \mathbb{N}$, for all $x, y \in \mathbb{R}^n_{++}$, for all $i, j \in \{1, \ldots, n\}$, and for all $\varepsilon \in \mathbb{R}_{++}$, if $x_i = y_i + \varepsilon \leq y_j - \varepsilon = x_j < y_j \leq z$ and $x_k = y_k$ for all $k \in \{1, \ldots, n\} \setminus \{i, j\}$, then $(z; y)P(z; x)$.

All of the properties introduced so far apply to each poverty line but no restrictions are imposed across different lines. The final two axioms differ in that regard.

The next axiom is an independence condition with respect to the poverty line. Suppose that, in two distributions x and y of the same population size, everyone is poor for a given poverty line z. Now suppose that the poverty line is increased to $z' > z$. Everyone remains poor under the new poverty line, and it seems reasonable to require that the relative ranking of x and y is unchanged if the new poverty line z' rather than the original z applies. The following property captures this requirement.

Poverty-line independence. For all $z, z' \in \mathbb{R}_{++}$ such that $z < z'$, for all $n \in \mathbb{N}$, and for all $x, y \in \mathbb{R}^n_{++}$, if $x_i \leq z$ and $y_i \leq z$ for all $i \in \{1, \ldots, n\}$, then

$$(z; x)R(z; y) \;\Leftrightarrow\; (z'; x)R(z'; y).$$

Combined with extended focus, poverty-line independence has the practically relevant implication that changes in the poverty line do not influence poverty assessments as long as the poor and the non-poor remain the same.

Finally, scale invariance is a common property that requires a poverty ordering to be relative. It requires that poverty comparisons are unchanged if both the poverty line and the fixed-population income distributions under consideration are multiplied by a common positive constant.

Scale invariance. For all $z \in \mathbb{R}_{++}$, for all $n \in \mathbb{N}$, for all $x, y \in \Omega$, and for all $\lambda \in \mathbb{R}_{++}$,

$$(\lambda z; \lambda x)R(\lambda z; \lambda y) \;\Leftrightarrow\; (z; x)R(z; y).$$

With the exception of extended focus and poverty-line independence, all of the above axioms are well-established in the literature on poverty measurement. For example, the Watts ordering (Watts, 1968) and the Foster-Greer-Thorbecke orderings with a parameter value above one (Foster, Greer, and Thorbecke, 1984) satisfy them. Moreover, the Watts ordering also satisfies poverty-line independence.

4 Characterization results

Our main result characterizes the class of generalized poverty-gap orderings.

Theorem 1. *A poverty ordering R satisfies anonymity, extended focus, strict monotonicity below and at the poverty line, fixed-number independence, continuity below and at the poverty line, and poverty-line independence if and only if R is a generalized poverty-gap ordering.*

The theorem is proven by employing some observations from the literature on population ethics and adapting them to the poverty framework considered here; see also Huseby (2012) and Bossert, Cato, and Kamaga (2020a,b) for some links between population ethics and sufficientarian theories of justice. As a first step, we focus on income distributions such that everyone's income is below or at the poverty line. The following auxiliary result shows that the generalized poverty-gap criterion must apply in these situations.

Lemma 1. *If a poverty ordering R satisfies anonymity, extended focus, strict monotonicity below and at the poverty line, fixed-number independence, and continuity below and at the poverty line, then, for all $z \in \mathbb{R}_{++}$, there exists a continuous and increasing function $g^z\colon (0, z] \to \mathbb{R}$ such that, for all $x, y \in \Omega$,*

$$(z; x)R(z; y) \;\Leftrightarrow\; \sum_{i \in \mathcal{P}(z;x)} (g^z(z) - g^z(x_i)) \geq \sum_{i \in \mathcal{P}(z;y)} (g^z(z) - g^z(y_i))\,. \tag{1}$$

Proof. Because none of the axioms in the lemma statement involve any changes in the poverty line, we can, without loss of generality, assume that $z \in \mathbb{R}_{++}$ is fixed. Setting $y = (z) \in \mathbb{R}_{++}$ in the first part of extended focus, it follows that $(z; x)I(z; (x, z))$ for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}^n_{++}$ (and, thus, for all $x \in (0, z]^n$). This means that z is a fixed critical level for R (see Blackorby and Donaldson, 1984). Combined with the axioms of anonymity, strict monotonicity below and at the poverty line, fixed-number independence, and continuity below and at the poverty line, this implies that there exists a continuous and increasing function $g^z\colon (0, z] \to \mathbb{R}$ such that

$$(z; x)R(z; y) \;\Leftrightarrow\; \sum_{i \in \mathcal{P}(z;x)} (g^z(x_i) - g^z(z)) \leq \sum_{i \in \mathcal{P}(z;y)} (g^z(y_i) - g^z(z)) \tag{2}$$

for all $x, y \in \Omega$. This observation is an immediate consequence of adapting the requisite results of Blackorby and Donaldson (1984) and Blackorby, Bossert, and Donaldson (2005, Chapters 4 and 6) to our domain. Note that a less-than-or-equal-to inequality is obtained in (2) because the requisite monotonicity property requires decreasingness rather than increasingness. Clearly, (2) is equivalent to (1). ■

We now use this lemma to prove our main characterization theorem.

Proof of Theorem 1. That the generalized poverty-gap orderings satisfy all of the axioms is straightforward to verify.

Conversely, suppose that R is a poverty ordering that satisfies the axioms. Let $z \in \mathbb{R}_{++}$, $n, m \in \mathbb{N}$, $x \in \mathbb{R}^n_{++}$, and $y \in \mathbb{R}^m_{++}$. We can, without loss of generality, assume that both $\mathcal{P}(z; x)$ and $\mathcal{P}(z; y)$ are non-empty; this is the case because $(z; x)I(z; (x, z))$ by extended focus and, thus, $(z; x)R(z; y)$ if and only if $(z; (x, z))R(z; y)$ by transitivity. Of course, the same argument applies to y. Using our notational convention, we write x and y as $x = (x^p, x^{np})$ and $y = (y^p, y^{np})$. Note that x^{np} or y^{np} may be the empty vector, which is the case if everyone is poor in the requisite distribution. By extended focus and transitivity, $(z; x)R(z; y)$ if and only if $(z; x^p)R(z; y^p)$ and, by Lemma 1, it follows that (1) is true for all $z \in \mathbb{R}_{++}$ and for all $x, y \in \Omega$.

We now use poverty-line independence to show that the functions g^z can be chosen to be the same for all poverty lines $z \in \mathbb{R}_{++}$. To this end, let $z, z' \in \mathbb{R}_{++}$ be two poverty lines such that $z < z'$. Let $x \in (0, z]^2$ and define $a, b \in \mathbb{R}_{++}$ by

$$a = (g^z)^{-1}\left(\frac{g^z(x_1) + g^z(x_2)}{2}\right) \quad \text{and} \quad b = (g^{z'})^{-1}\left(\frac{g^{z'}(x_2) + g^{z'}(x_2)}{2}\right),$$

that is, a and b are the quasilinear means of x generated by g^z and $g^{z'}$, respectively. These are well-defined since g^z and $g^{z'}$ are continuous and increasing on $(0, z]$. Furthermore, $a, b \in (0, z]$. Using (1), we obtain

$$(z; (a, a))I(z; x)$$

and

$$(z'; (b, b))I(z'; x).$$

By poverty-line independence, $(z; (a, a))I(z; x)$ implies that

$$(z'; (a, a))I(z'; x).$$

By the transitivity of R, we obtain

$$(z'; (a, a))I(z'; (b, b)).$$

This implies $a = b$ since R satisfies strict monotonicity below and at the poverty line. Thus, we obtain that, for all $x \in (0, z]^2$,

$$(g^z)^{-1}\left(\frac{g^z(x_1) + g^z(x_2)}{2}\right) = (g^{z'})^{-1}\left(\frac{g^{z'}(x_2) + g^{z'}(x_2)}{2}\right).$$

From Theorem 83 of Hardy, Littlewood, and Pólya (1934; 1952, p. 66), it follows that g^z and the restriction of $g^{z'}$ to $(0, z]$ must be increasing affine transformations of each other. Because the function g^z in (1) is unique up to increasing affine transformations, g^z and $g^{z'}$ can, without loss of generality, be chosen to be the same on $(0, z]$. Since this applies to all $z, z' \in \mathbb{R}_{++}$ with $z < z'$, there exists a continuous and increasing function $g\colon \mathbb{R}_{++} \to \mathbb{R}$ such that, for all $z, t \in \mathbb{R}_{++}$, $g^z(t) = g(t)$. Therefore, R must be a generalized poverty-gap ordering. ■

If the principle of transfers below and at the poverty line is added to the axioms of Theorem 1, the function g must be strictly concave. This is an immediate consequence of the well-known result that the principle of transfers is equivalent to strict concavity in the context of an additive criterion. Thus, we obtain the following result which we state without a proof.

Theorem 2. *A poverty ordering R satisfies anonymity, extended focus, strict monotonicity below and at the poverty line, fixed-number independence, continuity below and at the poverty line, poverty-line independence, and the principle of transfers below and at the poverty line if and only if R is a generalized poverty-gap ordering with a strictly concave function g.*

The final result of this section illustrates the consequences of adding scale invariance to the axioms of Theorem 2. The resulting characterization identifies the subclass of generalized poverty-gap orderings that are based on means of order r, where the parameter r must be such that the function g is increasing and strictly concave.

Theorem 3. *A poverty ordering R satisfies anonymity, extended focus, strict monotonicity below and at the poverty line, fixed-number independence, continuity below and at the poverty line, poverty-line independence, the principle of transfers below and at the poverty line, and scale invariance if and only if R is a generalized poverty-gap ordering such that either there exists $r \in (0, 1)$ such that $g(t) = t^r$ for all $t \in \mathbb{R}_{++}$, or $g(t) = \ln(t)$ for all $t \in \mathbb{R}_{++}$.*

Proof. As established in Theorem 2, all generalized poverty-gap orderings with a strictly concave function g (and, thus, the subclass identified in the statement of Theorem 3) satisfy all of the axioms except for scale invariance. Moreover, it is straightforward to verify that the orderings based on the functions g of the statement satisfy scale invariance.

To prove the converse implication, note first that R must be a generalized poverty-gap ordering with a strictly concave function $g\colon \mathbb{R}_{++} \to \mathbb{R}$ by virtue of Theorem 2. Let $z \in \mathbb{R}_{++}$ and $x \in (0, z]^2$. Let $a \in (0, z]$ denote the quasilinear mean of x generated by g, that is,

$$a = g^{-1}\left(\frac{g(x_1) + g(x_2)}{2}\right).$$

Since R is the generalized poverty-gap ordering associated with g, we obtain

$$(z; x)I(z; (a, a)).$$

From scale invariance, it follows that, for all $\lambda \in \mathbb{R}_{++}$,

$$(\lambda z; \lambda x)I(\lambda z; \lambda(a, a)).$$

Thus, we obtain that

$$g^{-1}\left(\frac{g(\lambda x_1) + g(\lambda x_2)}{2}\right) = \lambda a = g^{-1}\left(\frac{g(x_1) + g(x_2)}{2}\right)\lambda$$

for all $x \in (0, z]^2$ and for all $\lambda \in \mathbb{R}_{++}$. Since $z \in \mathbb{R}_{++}$ was chosen arbitrarily, we obtain that, for all $x \in \mathbb{R}^2_{++}$ and for all $\lambda \in \mathbb{R}_{++}$,

$$g^{-1}\left(\frac{g(\lambda x_1) + g(\lambda x_2)}{2}\right) = g^{-1}\left(\frac{g(x_1) + g(x_2)}{2}\right)\lambda.$$

Applying the second part of Theorem 2 in Section 3.1.3 of Aczél (1966, p. 153), the solution to this functional equation is given by

$$g(t) = \alpha \ln(t) + \beta \ \text{ or } \ g(t) = \alpha t^r + \beta$$

for some $\alpha, r \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$. Since g is strictly concave, r must be less than one. Furthermore, by the definition of a generalized poverty-gap ordering, we can, without loss of generality, assume that $\alpha = 1$ and $\beta = 0$ so that the functions of the theorem statement result. ■

5 Inequality among the poor

Suppose that the absence of poverty is identified with the welfare of those below and at the poverty line. As mentioned earlier, this is an interpretation that is well in line with our approach, and it allows us to think of R as a reverse welfare ordering defined for the poor.

Consider a poverty line $z \in \mathbb{R}_{++}$ and an income distribution $x \in \Omega$. An income level $\xi^p \in \mathbb{R}_{++}$ is an equally-distributed-equivalent income of the poor for z and x if

$$\left(z; \xi^p \mathbf{1}^{|\mathcal{P}(z;x)|}\right) I\left(z; x^p\right). \tag{3}$$

That is, if everyone among the poor has the same income ξ_p, the resulting equal distribution among the poor is associated with the same level of poverty as the restriction of the original distribution x to those who are poor.

If $\mathcal{P}(z; x) = \varnothing$, there is no well-defined equally-distributed-equivalent income of the poor for z and x. We therefore only consider poverty lines $z \in \mathbb{R}_{++}$ and distributions $x \in \Omega$ such that $\mathcal{P}(z; x) \neq \varnothing$ in the remainder of this section. In this case, the existence of a unique equally-distributed-equivalent income level for the poor is guaranteed whenever the ordering R satisfies extended focus, strict monotonicity below and at the poverty line, and continuity below and at the poverty line. Because of this uniqueness, we can express ξ^p as a function $\Xi^p(z; x)$ that depends on the poverty line z and the distribution under consideration x.

If R is a generalized poverty-gap ordering with a function g, (3) can be rewritten as

$$|\mathcal{P}(z; x)| \left(g(z) - g(\xi^p)\right) = \sum_{i \in \mathcal{P}(z;x)} \left(g(z) - g(x_i)\right)$$

or, solving for ξ^p and using the uniqueness of the equally-distributed-equivalent income,

$$\xi^p = \Xi^p(z; x) = g^{-1}\left(\frac{1}{|\mathcal{P}(z; x)|} \sum_{i \in \mathcal{P}(z;x)} g(x_i)\right) \tag{4}$$

for all $z \in \mathbb{R}_{++}$ and for all $x \in \Omega$ such that $\mathcal{P}(z;x) \neq \varnothing$. Let

$$\mu^p(z;x) = \frac{1}{|\mathcal{P}(z;x)|} \sum_{i \in \mathcal{P}(z;x)} x_i$$

be the arithmetic mean of the incomes of the poor.

The Atkinson-Kolm-Sen ethical index of inequality for the poor is defined as

$$I^p_{AKS}(z;x) = \frac{\mu^p(z;x) - \Xi^p(z;x)}{\mu^p(z;x)} = 1 - \frac{\Xi^p(z;x)}{\mu^p(z;x)}$$

for all $z \in \mathbb{R}_{++}$ and for all $x \in \Omega$ such that $\mathcal{P}(z;x) \neq \varnothing$; see Kolm (1969), Atkinson (1970), and Sen (1973). This index measures the percentage income shortfall of the poor that can be attributed to inequality among the poor.

The Atkinson-Kolm-Sen index is relative if it is invariant with respect to equal proportional changes in all incomes, that is, if $I^p_{AKS}(\lambda z; \lambda x) = I^p_{AKS}(z;x)$ for all $z \in \mathbb{R}_{++}$, for all $x \in \Omega$, and for all $\lambda \in \mathbb{R}_{++}$. Because the arithmetic mean μ^p is homogeneous of degree one, it follows immediately that the Atkinson-Kolm-Sen index is relative if and only if Ξ^p is homogeneous of degree one—or, equivalently, the ordering R is homothetic.

The ethical inequality index I^p_{AKS} is normatively significant in the sense that, for a given level of the average income of the poor, a distribution exhibits more poverty (equivalently, less welfare for the poor) than another if and only if the former is more unequal than the latter. This is true for any choice of a poverty ordering R.

An alternative ethical index of inequality is explored in Blackorby and Donaldson (1978). They use the transformation function, which is a homogeneous implicit representation of R, to define an ethical index of inequality. In contrast to the Atkinson-Kolm-Sen index, the index of Blackorby and Donaldson is always a relative index (because the transformation function is homogeneous of degree one) but it may depend on a reference level of the transformation function unless R is homothetic.

Using (4), the Atkinson-Kolm-Sen index for the generalized poverty-gap ordering corresponding to a function g is given by

$$I^p_{AKS}(z;x) = 1 - \frac{g^{-1}\left(\frac{1}{|\mathcal{P}(z;x)|} \sum_{i \in \mathcal{P}(z;x)} g(x_i)\right)}{\mu^p(z;x)}$$

for all $z \in \mathbb{R}_{++}$ and for all $x \in \Omega$ such that $\mathcal{P}(z;x) \neq \varnothing$. Following Kolm (1976), a strictly inequality-averse index of this form is relative if the function g is given by $g(t) = t^r$ for all $t \in \mathbb{R}_{++}$, where $r \in (0,1)$ is a parameter. For the parameter value $r = 0$, the corresponding function g is given by $g(t) = \ln(t)$ for all $t \in \mathbb{R}_{++}$. See also our Theorem 3 in the previous section.

The counterpart of the Atkinson-Kolm-Sen index is based on absolute rather than percentage income shortfalls among the poor that are attributable to inequality. The resulting ethical index is the Kolm index defined by

$$I^p_K(z;x) = \mu^p(z;x) - \Xi^p(z;x)$$

for all $z \in \mathbb{R}_{++}$ and for all $x \in \Omega$ such that $\mathcal{P}(z;x) \neq \varnothing$; see Kolm (1969, 1976) and Pollak (1971).

The Kolm index is absolute if it is invariant with respect to equal absolute changes in all incomes, that is, if $I^p_K(z+\delta; x+\delta) = I^p_K(z;x)$ for all $z \in \mathbb{R}_{++}$, for all $n \in \mathbb{N}$, for $x \in \mathbb{R}^n_{++}$, and for all $\delta \in \mathbb{R}$ such that $(z+\delta; x+\delta) \in \mathbb{R}_{++} \times \mathbb{R}^n_{++}$. Because the arithmetic mean μ^p is unit translatable, it follows immediately that the Kolm index is absolute if and only if Ξ^p is unit translatable—or, equivalently, the ordering R is translatable. As is the case for the Atkinson-Kolm-Sen index, the ethical inequality index I^p_K is always normatively significant.

The absolute counterpart of Blackorby and Donaldson's (1978) approach in terms of the transformation function is analyzed in Blackorby and Donaldson (1980a). They define a translation function that parallels the transformation function and obtain an ethical index that is always absolute but, again, their index may depend on a reference level of the translation function unless R is translatable.

The Kolm index for a generalized poverty-gap ordering corresponding to a function g is defined by

$$I^p_K(z;x) = \mu^p(z;x) - g^{-1}\left(\frac{1}{|\mathcal{P}(z;x)|}\sum_{i\in\mathcal{P}(z;x)} g(x_i)\right)$$

for all $z \in \mathbb{R}_{++}$ and for all $x \in \Omega$ such that $\mathcal{P}(z;x) \neq \varnothing$. A strictly inequality-averse index of this form is absolute if the function g is given by $g(t) = -\exp(-\gamma t)$ for all $t \in \mathbb{R}_{++}$, where $\gamma \in \mathbb{R}_{++}$ is a parameter; see Kolm (1976).

The approach outlined in this section differs from Blackorby and Donaldson's (1980b) examination of the possible links between welfare, poverty, and inequality among the poor. In their contribution, Blackorby and Donaldson define a relative poverty index as a function of the head-count ratio and the Atkinson-Kolm-Sen index for the poor. In addition, they perform an analogous analysis for an absolute notion of welfare and poverty. Our relationship between poverty and inequality is similar but deviates from Blackorby and Donaldson's (1980b) formulation because our poverty measures do not depend on the head-count ratio and, therefore, it is possible to interpret them as inverse welfare indicators for the poor. See also Vaughan (1987) for some links between welfare and poverty. Bossert (1990b) explores the relationships between Blackorby and Donaldson's (1980b) classes of poverty measures and the replication-invariance principle.

6 An application

We apply one of the scale-invariant generalized poverty-gap orderings identified in Theorem 3 to statistical data on income distributions in the United States. Our main purpose is to compare the trend in poverty between 1991 and 2017 with the trends that are obtained when using some standard relative ordinal indexes of poverty. Recall that a fixed poverty line is employed in the USA and, thus, this constitutes a natural practical application of the approach to poverty measurement that we advocate here. Because one of the comparison measures is given by the Watts (1968) ordering that is based on a logarithmic criterion, it is natural to choose the generalized poverty-gap ordering that corresponds the function defined by $g(t) = \ln(t)$ for all $t \in \mathbb{R}_{++}$. We denote this ordering by $R_{\ln}$.

The poverty orderings with which we compare our logarithmic ordering are two members of the class of Foster-Greer-Thorbecke orderings (Foster, Greer and Thorbecke, 1984) and the Watts ordering (Watts, 1968). The Foster-Greer-Thorbecke ordering R^{α}_{FGT} is defined by letting, for all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n_{++}$, and for all $y \in \mathbb{R}^m_{++}$,

$$(z;x) R^{\alpha}_{FGT} (z;y) \; \Leftrightarrow \; \frac{1}{n} \sum_{i \in \mathcal{P}(z;x)} \left(\frac{z - x_i}{z} \right)^{\alpha} \geq \frac{1}{m} \sum_{i \in \mathcal{P}(z;y)} \left(\frac{z - y_i}{z} \right)^{\alpha}$$

where $\alpha \in \mathbb{R}_+$ is a parameter. For $\alpha = 0$ and $\alpha = 1$, we obtain the head-count ratio and the poverty-gap ordering, respectively. If $\alpha = 2$, the squared poverty-gap ordering results. Unlike the head-count ratio and the poverty-gap ordering, the latter is distributionally sensitive. In general, a Foster-Greer-Thorbecke ordering satisfies the principle of transfers below and at the poverty line if and only if $\alpha > 1$. For our application, we employ the head-count ratio ordering R^0_{FGT} and the squared poverty-gap ordering R^2_{FGT}.

The Watts ordering R_W is closely related to the logarithmic generalized poverty-gap ordering $R_{\ln}$. It is defined by letting, for all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n_{++}$, and for all $y \in \mathbb{R}^m_{++}$,

$$(z;x) R_W (z;y) \; \Leftrightarrow \; \frac{1}{n} \sum_{i \in \mathcal{P}(z;x)} (\ln(z) - \ln(x_i)) \geq \frac{1}{m} \sum_{i \in \mathcal{P}(z;y)} (\ln(z) - \ln(y_i)) \,.$$

That is, the Watts ordering is the per-capita counterpart of the logarithmic generalized poverty-gap ordering. Because of this analogy, the logarithmic generalized poverty-gap ordering seems to be a suitable choice for the comparisons carried out in this section. See Zheng (1993) for an axiomatization of a representation of the Watts index.

Although we only consider ordinal comparisons, we perform our computations using the specific representations of $R_{\ln}$, R^0_{FGT}, R^2_{FGT}, and R_W given by

$$\sum_{i \in \mathcal{P}(z;x)} (\ln(z) - \ln(x_i)) \,,$$

$$\frac{|\mathcal{P}(z;x)|}{n},$$

$$\frac{1}{n} \sum_{i \in \mathcal{P}(z;x)} \left(\frac{z - x_i}{z} \right)^2,$$

and

$$\frac{1}{n} \sum_{i \in \mathcal{P}(z;x)} (\ln(z) - \ln(x_i)) \,.$$

To compare the logarithmic generalized poverty-gap ordering with those of Foster, Greer, and Thorbecke (1984) and Watts (1968), we measure national poverty in the United States over the period from 1991 to 2017. We use Waves III to X of the Luxembourg Income

Study (LIS) datasets, obtained from the Current Population Survey—Annual Social and Economic Supplement, referred to as the March Supplement until 2002. These datasets are used by the United States Bureau of Census for estimating the official poverty rate, which is given by the head-count ratio R^0_{FGT}. Although the official poverty rate of the United States is calculated using gross incomes, we employ disposable household cash incomes and augment them by including in-kind food benefits (such as food stamps) to obtain measures that better reflect actual poverty. See Notten and de Neubourg (2011) and Smeeding (2006) for the use of disposable cash income in measuring poverty in the United States. These household incomes are equivalized by means of the square root equivalence scale; therefore, the unit of analysis is the individual. When calculating the poverty measures, missing and negative incomes are excluded. In addition, zero incomes are excluded in the calculation of $R_{\ln}$ and R_W.

The official poverty line of the United States is updated annually, using the Consumer Price Index for all Urban Consumers. Strictly speaking, there is a set of poverty lines, each of which defines the poverty threshold for a specific type of family categorized by size and composition. Since our primary purpose is the comparison of the four poverty measures, we use, for simplicity, the weighted average of these poverty thresholds for a family of four. This figure, which is published by the United States Bureau of Census, is equivalized by means of the square root equivalence scale. As pointed out by numerous authors, the official poverty line of the United States is rather low compared to median income. Between 1991 and 2017, the ratio of the official poverty line to the median equivalized disposable cash income varies between 34 and 42 per cent. Therefore, in addition, we report the results that are obtained for the line that is given by 125 per cent of the official poverty line. For the latter, the corresponding ratio lies between 43 and 52 per cent. We note that the use of this alternative (higher) poverty line is standard practice in the requisite literature.

The trends in poverty in the United States according to the four poverty measures are summarized in Figure 1, where the poverty line employed is the official poverty threshold of the United States. We normalize the values of the poverty measures calculated for each year between 1991 and 2017 by dividing them by the corresponding 1991 values.

As Figure 1 shows, poverty in the United States is increasing, in particular, from 2000 onward, if it is evaluated by $R_{\ln}$. By definition, the trends in poverty as measured by $R_{\ln}$ and R_W fluctuate synchronously. The fluctuations according to R_W are weaker in magnitude reflecting the population growth in the United States—total population has been growing at annual rates of about 1.4 to 0.6 per cent between 1991 and 2017, from 255 million in 1991 to 324 million in 2017. While the poverty comparisons by $R_{\ln}$ and R_W are the same when comparing adjacent periods, their comparisons for more distant periods may differ. For instance, $R_{\ln}$ concludes that poverty in 2006 is higher than in 1991, whereas R_W reaches the opposite conclusion. The increasing trend in poverty according to $R_{\ln}$ is no longer observed if R^2_{FGT} is employed. Furthermore, the trend in poverty as measured by the head-count ratio is decreasing over the sampling period; the corresponding official poverty rates do not exhibit a clear decreasing trend (Semega, Kollar, Shrider, and Creamer, 2020).

Figure 2 summarizes the trends in poverty measured by the four measures setting the poverty line at 125 per cent of the official poverty threshold. As the figure shows, the trends are overall the same as those evaluated for the official poverty threshold. Thus,

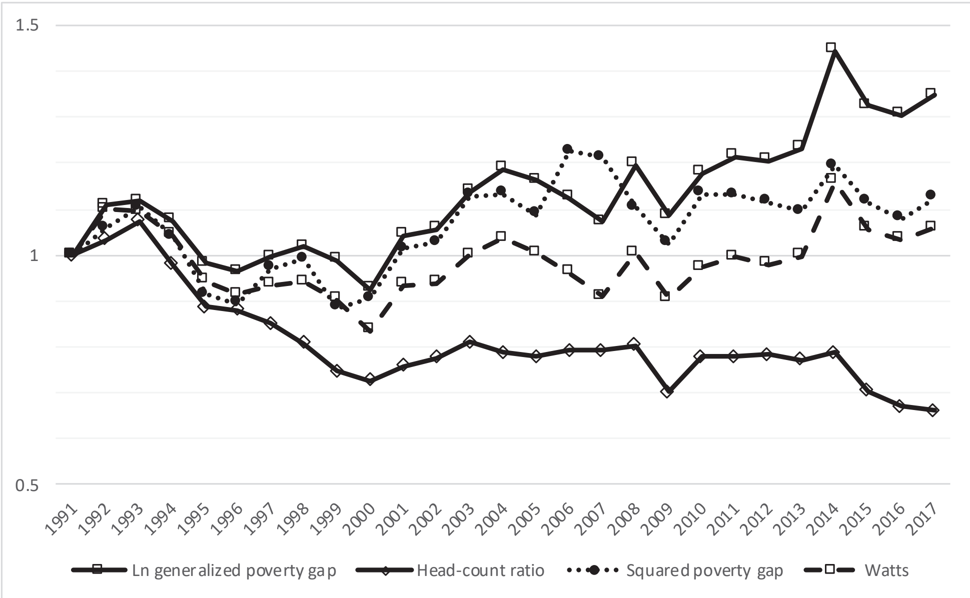

Figure 1: Trends in poverty in the United States over the period 1991–2017

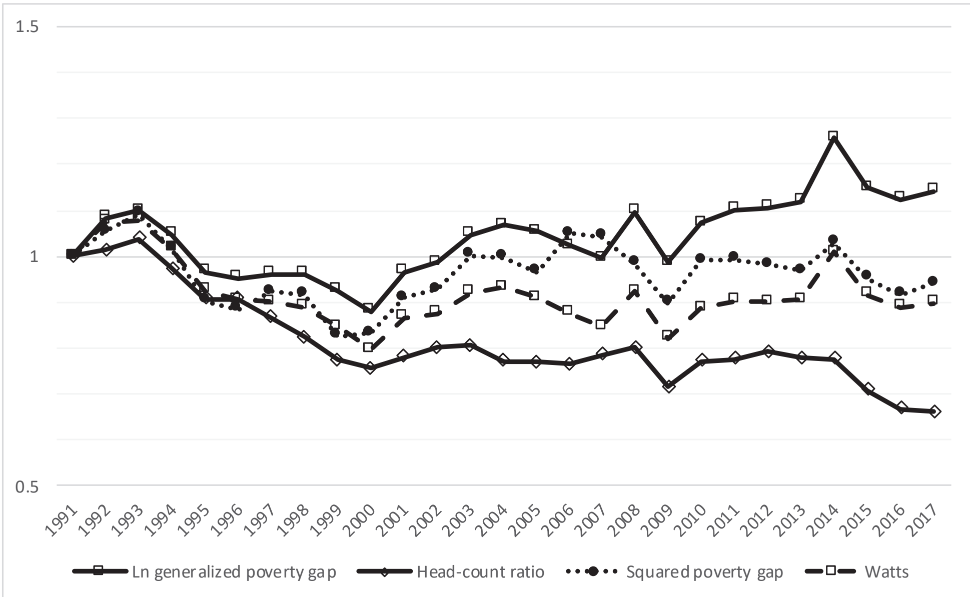

Figure 2: Trends in poverty in the United States over the period 1991–2017 (125% of the official poverty threshold)

the observations for the official poverty threshold are robust in this respect. However, a major difference between the official poverty threshold and its 125 percent augmentation is that, in the latter case, poverty comparisons that involve distant periods according to R_W and R^2_{FGT} lead to the conclusion that recent levels of poverty in the United States have decreased as compared to the early 1990s.

7 Conclusion

As mentioned earlier, an important feature of our class of poverty orderings is that they do not depend on the head-count ratio—or, more generally, on the overall size of the population or the number of those below or at the poverty line. This means that a given aggregate generalized poverty gap is equally (un)desirable, independent of how many people there are in total, and independent of how many individuals find themselves below or at the poverty line. Of course, the ceteris-paribus addition of a poor person increases poverty because it increases the aggregate generalized income gap but there is no additional increase in poverty. More importantly, our measures are insensitive with respect to the existence of non-poor people, a property that we consider essential. It seems questionable to claim that a reduction in poverty has occurred when all that took place is an increase in the number of non-poor and the situation of the poor has not changed in any way.

Appendix: Independence of the axioms of Theorems 1, 2, and 3

The generalized poverty-gap ordering associated with the transformation $g(t) = t$ for all $t \in \mathbb{R}_{++}$ satisfies all of our axioms of Theorems 2 and 3 except the principle of transfers below and at the poverty line. If a generalized poverty-gap ordering is associated with the transformation $g(t) = -\exp(-t)$ for all $t \in \mathbb{R}_{++}$, it satisfies all of our axioms of Theorem 3 except scale invariance. Furthermore, the Watts ordering R_W satisfies all of our axioms of Theorems 1, 2, and 3 except extended focus. We show that each of the other five axioms of Theorems 1, 2, and 3 is independent.

We define the poverty ordering R^1 as follows. For all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n_{++}$, and for all $y \in \mathbb{R}^m_{++}$,

$$(z;x)R^1(z;y) \;\Leftrightarrow\; \sum_{i \in \mathcal{P}(z;x)} (\ln(\alpha_i) - \ln(x_i)) \geq \sum_{i \in \mathcal{P}(z;y)} (\ln(\alpha_i) - \ln(y_i)),$$

where $\alpha_1 = 2z$ and $\alpha_j = z$ for all $j \in \mathbb{N} \setminus \{1\}$. The poverty ordering R^1 satisfies all axioms of Theorems 1, 2, and 3 except anonymity. Letting $z \in \mathbb{R}_{++}$, consider $x = (2z, z/2)$ and $y = (z/2, 2z)$. Then, yP^1x follows since $\mathcal{P}(z;x) = \{2\}$, $\mathcal{P}(z;y) = \{1\}$, and $\ln(\alpha_2) - \ln(x_2) = \ln(z) - \ln(z/2) < \ln(2z) - \ln(z/2) = \ln(\alpha_1) - \ln(y_1)$.

Hereafter, for any $z \in \mathbb{R}_{++}$, for any $n \in \mathbb{N}$, and for any $x \in \mathbb{R}^n_{++}$, let $x^* = (x^*_1, \ldots, x^*_n)$ denote the censored income distribution corresponding to x defined by, for each $i \in \{1, \ldots, n\}$,

$$x^*_i = \min\{x_i, z\}.$$

Define the poverty ordering R^2 by letting, for all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n_{++}$, and for all $y \in \mathbb{R}^m_{++}$,

$$(z;x)R^2(z;y) \Leftrightarrow \sum_{i=1}^{n}(z^2 - (x_i^*)^2) \leq \sum_{i=1}^{m}(z^2 - (y_i^*)^2).$$

The poverty ordering R^2 satisfies all axioms of Theorems 1, 2, and 3 except strict monotonicity below and at the poverty line.

For $n \in \mathbb{N}$ and for $x \in \mathbb{R}^n_{++}$, let $(x_{[1]}, \ldots, x_{[n]})$ denote a permutation of x such that $x_{[1]} \leq \cdots \leq x_{[n]}$; that is, the incomes in such a permutation are ranked from lowest to highest, with ties being broken arbitrarily. We define the poverty ordering R^3 as follows. For all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n_{++}$, and for all $y \in \mathbb{R}^m_{++}$,

$$(z;x)R^3(z;y) \Leftrightarrow \sum_{i=1}^{n} \frac{1}{i}\left(z - x^*_{[i]}\right) \geq \sum_{i=1}^{m} \frac{1}{i}\left(z - y^*_{[i]}\right).$$

That is, the poverty ordering R^3 applies the inequality-loving *generalized Gini ordering* with the weights $1/i$ to the shortfalls of censured incomes from a poverty line; see Gini (1912), Mehran (1976), Donaldson and Weymark (1980), Weymark (1981), and Bossert (1990a) for generalizations of the Gini ordering. The poverty ordering R^3 satisfies all axioms of Theorems 1, 2, and 3 except fixed-number independence.

For all $n \in \mathbb{N}$, the *leximin ordering* R_{n-lex} on $\mathbb{R}^n$ is defined by letting, for all $x, y \in \mathbb{R}^n$,

$$\begin{aligned} xR_{n-lex}y \Leftrightarrow\ & x \text{ is a permutation of } y \text{ or there exists } j \in \{1, \ldots, n\} \text{ such that} \\ & x_{[i]} = y_{[i]} \text{ for all } i < j \text{ and } x_{[j]} > y_{[j]}. \end{aligned}$$

We define the poverty ordering R^4 as follows. For all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n_{++}$, and for all $y \in \mathbb{R}^m_{++}$,

$$\begin{aligned} (z;x)R^4(z;y) \Leftrightarrow\ & [n = m \text{ and } (y_1^*, \ldots, y_n^*)R_{n-lex}(x_1^*, \ldots, x_n^*)] \text{ or} \\ & [n > m \text{ and } (y_1^*, \ldots, y_m^*, z\mathbf{1}_{n-m})R_{n-lex}(x_1^*, \ldots, x_n^*)] \text{ or} \\ & [n < m \text{ and } (y_1^*, \ldots, y_m^*)R_{m-lex}(x_1^*, \ldots, x_n^*, z\mathbf{1}_{m-n})]. \end{aligned}$$

That is, R^4 applies the critical-level leximin population principle of Blackorby, Bossert, and Donaldson (1996), in a reverse way, to censured income distributions using the poverty line z as a critical level. The poverty ordering R^4 satisfies all axioms of Theorems 1, 2, and 3 except continuity below and at the poverty line.

Finally, define the poverty ordering R^5 by letting, for all $z \in \mathbb{R}$ and for all $x, y \in \Omega$,

$$(z;x)R^5(z;y) \Leftrightarrow \sum_{i \in \mathcal{P}(z;x)} \left(\frac{z - x_i}{z}\right)^2 \geq \sum_{i \in \mathcal{P}(z;y)} \left(\frac{z - y_i}{z}\right)^2.$$

This is a variant of the squared poverty-gap ordering that measures poverty on a total basis. The poverty ordering R^5 satisfies all axioms of Theorems 1, 2, and 3 except poverty-line independence; for example, for $x = (9, 1)$ and $y = (4, 4)$, we obtain $(10;x)P^5(10;y)$ if $z = 10$, whereas $(15;y)P^5(15;x)$ follows if $z = 15$.

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