A Revealed Preference Test for Price Competition in Multi-products Differentiated Market (2nd Year Paper for Industrial Organization and Theory)

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Assumptions on competitive structure is often crucial to estimate marginal costs and to obtain counterfactual predictions. In this paper, tests for price competition among multi-products firms are introduced, which is based on firm's revealed preference (or, revealed profit). In contrast to other approaches based on estimated demand functions such as the conduct parameter estimation, it does not require any IVs. Simulations show that the competitive structure itself might put tight restrictions on data even if data is a small panel consisting of 6 products and 3 periods. In this paper, I employ a class of demand structure introduced by Nocke and Schutz (2016), discrete-continuous model, which nests the multinomial logit demand and CES demand functions.

1 Introduction

In the literature of Industrial Organization, we often assume specific competitive structures, such as price competition or quantity competition. Sometimes, such a competitive structure is crucial in empirical research. For instance, we often back out marginal cost from the first order conditions based on estimated demand functions. Results of counterfactual analysis, which often provides the main policy implication in research with structural models, also depends on the assumed competitive structures. Even though we can obtain estimates of parameters in a structural model, which fits to data the best, it is still possible that the structural model itself does not fit to the data. In other words, data might not be rationalized by any possible parameters. Furthermore, it might not be rationalized by any realizations of structural error terms. In this paper, I provide a systematic way to detect data which is inconsistent with price competition among single/multi-products firms under a class of demand structure introduced by Nocke and Schutz (2016).

For consumer's behavior, Afriat (1967) shows that finite data satisfies GARP if and only if it is rationalized by utility maximization given a price vector. In other words, if the data violates GARP, then it cannot be explained by any (locally non-satiated) utility functions. Brown and Matzkin (1996) extend this idea to the general equilibrium framework. Carvajal et al. (2013) apply the idea to Cournot competition, and show that the Cournot rationalizability can be checked by the existence of parameters which satisfy some conditions. Carvajal et al. (2014) introduced a few variants of Carvajal et al. (2013); a test for multi-products Cournot competition, and a test for a price competition in a differentiated market. However, for the price competition, they focus on a competition where each firm produces a single product, while price competition with multi-products firms are often examined in the empirical IO literature (e.g., Berry et al. (1995), Goldberg (1995)). One of the main difficulties to extend Carvajal et al. (2014)'s test to competition among multi-products firms arises from substitution effects among products produced by the same firm (or, cannibalization effects). We can circumvent such a difficulty by employing an important class of demand structure, discrete-continuous model introduced by Nocke and Schutz (2016), which nests the multinomial logit demand function and CES demand function as special cases.

In order to test the competitive structure, we can also estimate the *conduct parameter*. Bresnahan (1982) shows that we can identify the conduct parameter in an industry if there are rotation of demand functions over time. (See Bresnahan (1989) for estimations and its applications.) Alternatively, if we have data on cost structure, we could compare marginal costs backed out from the model and the actual cost data since different competitive structures give different FOCs, which returns different estimates of marginal costs (e.g., Wolfram (1999)). The revealed preference test examined in this paper provides an alternative approach with advantages and disadvantages. First advantage is the revealed preference test in this paper does not require any IVs while both the estimation of conduct parameter requires appropriate IVs and the approach by Wolfram (1999) also requires IVs to consistently estimate demand functions, which is used to estimate marginal costs. The reason why we don't need IVs is that demand functions are assumed to be know to each firm (but not to econometrician).[1] As long as the firms' maximize their own profits (known to themselves), it puts some data restrictions. It is analogous to Afriat (1967)'s theorem which characterize data restrictions satisfied as long as consumers maximize their own utility (known to themselves, but not known to the econometrician). Second, we only need market level price and quantity data, but not other characteristics, to implement the test. Therefore, this test can be used as a pretest/sanity-check before detailed estimation.

A disadvantage is that the test is a joint test of competitive structure and demand/cost functions. Therefore, rejection of the model might imply other types of competition under a discrete-continuous demand structure, price competition under other demand functions, or other competitions under other demand functions. Even though discrete-continuous model includes the logit demand function and CES demand function as special cases, it also has IIA property. Therefore, the main theorem in this paper does not hold for random coefficient logit model (e.g., Berry et al. (1995)). Another issue is that cost functions are assumed to be constant over time, which can be arbitrary convex functions. Therefore, the test should be implemented for short panel data where cost structure is not supposed to change during the range. In practice, if a researcher have a long panel data, then data can be cut into many short panels, the test can be implemented to each short data, and the rejection ratio can be reported. It is worth noting that even if data is as short as 3 periods, the model can put very tight restrictions especially when each firm produces many products. It is exemplified in simulations in Section 3.

The remaining of the paper is organized as follows. In section 2, I introduce the model and state the main results. In section 3, I examine the performance of the test introduced in the previous section. I summary in section 4.

2 The model

In this paper, I consider a standard framework for a competition in a differentiated market, where each firm produce different products. Each products can be similar but not completely same. We assume that demand functions can change over time, potentially because of change in consumer's taste or product's characteristics (which might be observed or unobserved by the econometrician). I

[1] In other words, this test is robust to an unobserved heterogeneity as long as demand is assumed in a class of discrete-continuous model.

denote $\mathcal{J} = \{1, 2, ..., J\}$ as a set of products and $Q_{j,t} : R_+^J \to R_+$ as a demand function of product $j \in \mathcal{J}$ at time $t \in \{1, ..., T\}$. The demand is assumed to be in a class of *discrete/continuous* model explained later. Firm f produces a set of products $\mathcal{J}_f \subset \mathcal{J}$ s.t. $\mathcal{J}_f \cap \mathcal{J}_g = \phi$ for $f \neq g$ and denote $J_f = |\mathcal{J}_f|$. A cost function of product $j \in \mathcal{J}$, $C_j : R_+ \to R$, is assumed to be convex and twice continuously differentiable. In this paper, I focus on time-invariant cost functions, which plays the similar role as time-invariant preference in Afriat (1967).[2] Then, the profit function for firm f at time t is written as $\pi_{f,t} = \sum_{j \in \mathcal{J}_f} \{Q_{j,t}(p) p_j - C_j(Q_{j,t}(p))\}$.

In the following, I mainly utilize the FOC of the profit functions and cost convexity to derive testable data restrictions, which should be satisfied regardless of the level of parameters and structural error terms. Using the profit function defined above, FOC w.r.t. p_j is written as

$$0 = Q_{j,t}(p) + \sum_{k \in J_f} \{p_k - C'_{k,t}(Q_{k,t}(p))\} \frac{\partial Q_{k,t}(p)}{\partial p_j}.$$

2.1 Example: Logit Demand Function

Before going to the main result with a general specification, I exemplify that some data cannot be explained by price competition with the logit demand function, which is a special case of the discrete/continuous model. By using the logit demand function, $Q_{j,t}(p) = M_t \frac{\exp(v_{jt} - \alpha p_{jt})}{1 + \sum_k \exp(v_{kt} - \alpha p_{kt})}$ for some M_t, $\alpha \in R_+$ and $(v_{j,t})_{j \in \mathcal{J}} \in R^J$, the first order condition is rewritten as;

$$0 = Q_{j,t}(p) - \{p_j - C'_{j,t}(Q_{j,t}(p))\} \alpha Q_{j,t}(p) + \sum_{k \in J_f} \{p_k - C'_{k,t}(Q_{k,t}(p))\} \frac{\alpha}{M_t} Q_{k,t}(p) Q_{j,t}(p)$$

By rearranging it, we obtain the following equation;

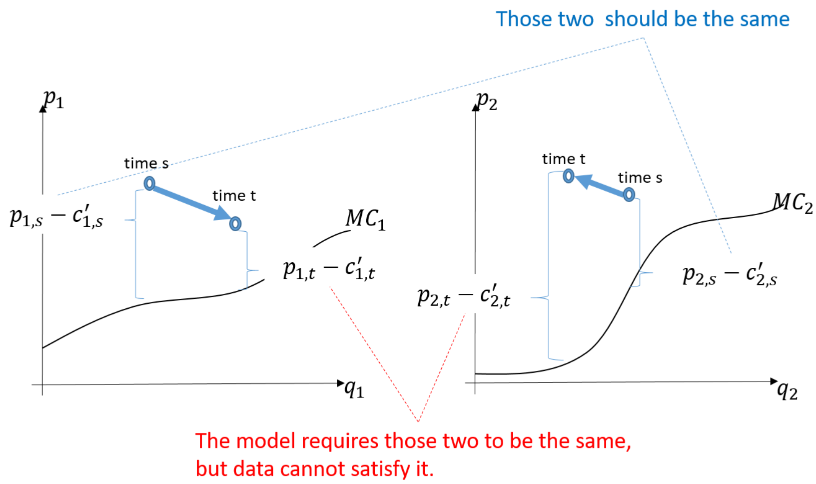
$$p_j - C'_{j,t}(Q_{j,t}(p)) = \alpha + \frac{1}{M_t} \sum_{k \in J_f} \{p_k - C'_{k,t}(Q_{k,t}(p))\} Q_{k,t}(p).$$

Here, RHS is common among goods produced by the same firm. Therefore,

$$p_j - C'_{j,t}(Q_{j,t}(p)) = p_k - C'_{k,t}(Q_{k,t}(p)) \quad (1)$$

[2] Carvajal et al. (2013) also provide a revealed preference test with a cost shifter common among all products, and Carvajal et al. (2014) provide a revealed preference test with some observed cost shifters. They are potential directions of the future research.

Figure 1: Example: Logit Demand Function



for any $j,\, k \in \mathcal{J}_f$. By combining with the increasing marginal cost assumption, it rejects the following data ; $(p_{j,\tau},\, q_{j,\tau})_{j=1,2,\,\tau=s,t}$ s.t. $\{1,2\} \subset \mathcal{J}_f$, $p_{1,s} > p_{1,t}$, $p_{2,s} < p_{2,t}$, $q_{1,s} < q_{1,t}$, and $q_{2,s} > q_{2,t}$ (see Fig. 1). Suppose that data satisfy eq.(1) at time s. Then, if marginal costs are increasing in own quantity, then $p_{1,t} - C'_{1,t}(q_{1,t}) < p_{1,s} - C'_{1,s}(q_{1,s}) = p_{2,s} - C'_{2,s}(q_{2,s}) < p_{2,t} - C'_{2,t}(q_{2,t})$. Therefore, eq.(1) is not satisfied at time t. Thus, this data $(p_{j,\tau},\, q_{j,\tau})_{j=1,2,\,\tau=s,t}$ cannot be explained by (a repetition of static) price competition under logit demand functions. It means that this data cannot be explained by any sets of parameters including demand parameters and non-parametric cost functions.

There are two important features of this result. First, the rejection or acceptance is not probabilistic even if the model has (only) the structural error term in the logit demand function. When we estimate logit demand functions from aggregate data, $v_{j,t}$ is decomposed to $v_{j,t} = x'_{j,t}\beta + \xi_{j,t}$ where $x_{j,t}$ is a vector of product j's observed characteristics, $\xi_{j,t}$ is unobserved characteristics, and β is a vector of parameters. In the logit demand estimation, $\xi_{j,t}$ is treated as a structural error term. However, eq.(1) should be satisfied regardless of the realization of $\xi_{j,t}$ as long as firms are competing in prices under logit demand functions. (Recall that I did not put any assumptions on $v_{j,t}$.) This is because the realization of $(\xi_{j,t})_{j\in\mathcal{J}}$ is assumed to be known to each firm (but not to the econometrician), which is often assumed in the literature of empirical IO.

Second, the above data is not rejected by the logit demand assumption alone, but it can be rejected if combined with price competition and cost convexity. For any data $(p_{j,t},\, q_{j,t})_{j=1,2}$ at each

t, we can back out the corresponding $(v_{j,t})_{j\in\mathcal{J}}$ by the inversion of market share function as in Berry (1994). If we only assume logit demand function, any changes in data over time can be captured by changes in $(v_{j,t})_{j\in\mathcal{J}}$ over time. On the the hand, if we also assume price competition and convex cost function, then eq.(1) and the increasing marginal cost provide the restrictions over time.

In the following part, I provide a set of inequalities as a systematic way to detect data inconsistent with price competitions, and show that such conditions are actually sufficient for rationalization by the price competition. Instead of the logit demand function, I employ a class of demand functions by Nocke and Schutz (2016), which nests the logit demand function and CES demand function.

2.2 Discrete-Continuous Demand Function

In the following, I employ the discrete-continuous demand function introduced by Nocke and Schutz (2016), where the demand function for product j is written as $Q_j(p) = m\frac{-h_j'(p_j)}{h_0+\sum_{k\in I} h_k(p_k)}$ where $h_j(\cdot)$ is decreasing and log-convex for every j and m is a positive constant. An important example of this demand function is the logit model, where $h_j(p_j) = \exp(v_j - \alpha p_j)$ and $m = M/\alpha$ where $v_j \in R$ is the value of goods j, $\alpha > 0$ is a coefficient for prices, $M > 0$ is the size of the market, and h_0 is the exponentiated value of the outside option. [3]

In this paper, I utilize the fact that we can express the partial derivatives of discrete-continuous demand function in a simple form;

$$\begin{aligned}
\frac{\partial Q_{j,t}(p)}{\partial p_j} &= m_t \frac{-h''_{j,t}(p_j)}{h_{0,t} + \sum_{k\in I} h_{k,t}(p_k)} + m_t \left(\frac{-h'_{j,t}(p_k)}{h_{0,t} + \sum_{k\in I} h_{k,t}(p_k)} \right)^2 \\
&= m_t \frac{-h''_{j,t}(p_k)}{h_{0,t} + \sum_{k\in I} h_{k,t}(p_k)} + m_t^{-1} (Q_{j,t}(p))^2 \\
&= Q_{j,t}(p) \left\{ \frac{-h''_{j,t}(p_k)}{-h'_{j,t}(p_k)} + m_t^{-1} Q_{j,t}(p) \right\} \\
&= -m_t^{-1} Q_{j,t}(p) \left\{ m_t \frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)} - Q_{j,t}(p) \right\}
\end{aligned}$$

[3] As discussed by Nocke and Schutz (2016), discrete-continuous model with the outside option can be normalized to discrete-continuous model without the outside option, $\tilde{Q}_j(p) = m\frac{-\tilde{h}_j'(p_j)}{\sum_{k\in I} \tilde{h}_k(p_k)}$, by letting $\tilde{h}_j(p_j) = \frac{1}{J} h_0 + h_j(p_j)$. In this paper, h_0 is explicitly denoted just to explain intuition of the results in later parts.

and

$$\begin{aligned}
\frac{\partial Q_{k,t}(p)}{\partial p_j} &= m\frac{-h'_{k,t}(p_k)}{\left(h_{0,t}+\sum_{k\in I}h_{k,t}(p_k)\right)^2}\left(-h'_{j,t}(p_j)\right). \\
&= m^{-1}Q_{k,t}(p)\cdot Q_{j,t}(p) \quad \forall k{\neq}j
\end{aligned}$$

It is worth noting that $m_t\frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)}-Q_{j,t}(p)$ is positive because of the log concavity of $h_j(\cdot)$. [4]

By using the above expression, the FOC is written as follows;

$$\begin{aligned}
0 &= 1+\sum_{k\in J_f}\left\{p_k-C'_{k,t}(Q_{k,t}(p))\right\}\frac{\partial Q_{k,t}(p)}{\partial p_j}\frac{1}{Q_{j,t}(p)} \\
&= 1-m_t^{-1}\left\{p_j-C'_{j,t}(Q_{j,t}(p))\right\}\left\{m_t\frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)}-Q_{j,t}(p)\right\} \\
&\quad +m_t^{-1}\sum_{k\in J_f,\,k\neq j}\left\{p_k-C'_{k,t}(Q_{k,t}(p))\right\}Q_{k,t}(p) \\
&= m_t-\left\{p_j-C'_{j,t}(Q_{j,t}(p))\right\}m_t\frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)}+\sum_{k\in J_f}\left\{p_k-C'_{k,t}(Q_{k,t}(p))\right\}Q_{k,t}(p)
\end{aligned}$$

Therefore, if the data $\{\bar{p},\,\bar{q}\}$ is generated by the price competition with (unknown) discrete-continuous demand function, there exists α_{jt}, δ_{jt} s.t.

$$0=m_t-\{\bar{p}_j-\delta_{j,t}\}\,m_t\alpha_{j,t}+\sum_{k\in J_f}\{\bar{p}_k-\delta_{j,t}\}\,\bar{q}_{k,t}, \tag{2}$$

which corresponds to $\frac{h''_{j,t}(\bar{p}_j)}{-h'_{j,t}(\bar{p}_j)}$ and $C'_j(\bar{q}_{j,t})$, respectively.

On the other hand, since $\delta_{j,t}$ corresponds to $C'_j(\bar{q}_{j,t})$ and $C'_j(\cdot)$ is assumed to be increasing, $\delta_{j,t}$ must be higher than $\delta_{j,s}$ $(s\neq t)$ if $\bar{q}_{j,t}$ is larger than $\bar{q}_{j,s}$. This is summarized as an inequality;

$$0\leq(\delta_{j,s}-\delta_{j,t})(\bar{q}_{j,s}-\bar{q}_{j,t}). \tag{3}$$

By combining eq.(2) and eq.(3), we can obtain a set of necessary conditions. Furthermore, it turns out to be also sufficient conditions for data to be rationalized by price competition. It is summarized in the following theorem.

[4]The log-concavity implies $\frac{h''_j(p)}{-h'_j(p)}>\frac{-h'_j(p)}{h_j(p)}\left(>\frac{-h'_j(p)}{h_0+\sum h_k(p)}\right)$.

Theorem1 (Discrete-Continuous): The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand rationalizable under convex cost function and discrete/continuous demand function if and only if there are real numbers $\alpha_{j,t}$, $\delta_{j,t}$, m_t for any $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_{j,t} + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{k,t}\} \bar{q}_{k,t}$; and
3. $0 \leq \left(\delta_{j,t'} - \delta_{j,t}\right)\left(\bar{q}_{j,t'} - \bar{q}_{j,t}\right)$.

The first set of conditions comes from underlying specifications of demand and cost functions; $\alpha_{j,t} > 0$ comes from an assumption that h_j is decreasing and log-convex, $\delta_{j,t} > 0$ comes from increasing cost functions, $m_t > 0$ comes from the assumption that quantity of each goods are non-negative. The proof of sufficiency consists of two steps. First, I re-construct demand and cost functions from $\alpha_{j,t}$ and $\delta_{j.t}$ which satisfies the conditions. Since the demand and cost functions are re-constructed to satisfy $\frac{h''_{j,t}(\bar{p}_j)}{-h'_{j,t}(\bar{p}_j)} = \alpha_{j,t}$ and $C'_j\left(\bar{q}_{j,t}\right) = \delta_{j.t}$, data $\bar{p}$ and $\bar{q}$ satisfies FOC under re-constructed demand and cost functions. In the second step, I show that the FOC is a sufficient condition for profit maximization given other firms' prices. It is not trivial since the profit function does not satisfy quasi-concavity. In this paper, the sufficiency is proved by the unique solution of FOC, which comes from the unique "common ι-markup" and a mapping from ι-markup to price vectors as in Nocke and Schutz (2016). See the Appendix for the full proof.

It is worth noting that the second condition is not linear in general because of an interaction of $\delta_{j,t}$ and $\alpha_{j,t}$ in contrast to Carvajal et al. (2013, 2014). It prevents us from using algorithms for linear programming. To implement the above test, we can consider an algorithm similar to moment inequalities. (See Section 3 for more details.)

Since the logit demand function can be represented as a special case of discrete-continuous choice where $\frac{h''_{j,t}(p_j)}{-h'_{j,t}(p_j)} = \frac{h''_{k,t}(p_k)}{-h'_{k,t}(p_k)} = \alpha_t$ for all $j, k \in \mathcal{J}$ and for all p_j, $p_k \in R_+$, the following corollary can be obtained easily.

Corollary1 (Logit): The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand rationalizable under convex cost function and logit demand function if and only if there are real numbers α_t, $\delta_{j,t}$, m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;

1. $\alpha_t > 0$, $\delta_{j,t} > 0$, $m_t > 0$;

2. $0 = m_t - \{\bar{p}_j - \delta_{j,t}\} m_t \alpha_t + \sum_{k \in J_f} \{\bar{p}_k - \delta_{j,t}\} \bar{q}_{k,t}$; and

3. $0 \leq (\delta_{j,t'} - \delta_{j,t}) (\bar{q}_{j,t'} - \bar{q}_{j,t})$.

It is worth noting that the above logit function allows the price coefficient α_t to vary over time. Another version of the corollary with constant α can be also obtained by replacing α_t to α for all t in the above statement.

In the logit specification, we can interpret the revealed preference test in comparison with an alternative procedure to check the competitive structure. Usually, a parameter of demand function, α, is estimated from aggregated data, and δ's are backed out from the first order condition. Potentially, we could check whether the obtained δ's are reasonable or not. Corollary 1 implies the similar procedure without the estimation of α. It check the monotonicity of δ recovered from any possible $\alpha > 0$. Potential advantage of such a test is that the test is robust to any endogenous quality since FOC should be still satisfied as long as firms know the quality even if it is unobserved to the econometrician.

As discussed in Subsection 2.1, the rejected data cannot be explained by a peculiar realization of some random components $\xi_{j,t}$, and any data can be rationalized if we only assume the logit demand function, but not price competition and increasing marginal costs.

2.3 More Strict Test with Additional Assumptions

Even though the above results provides some non-trivial constraints, the rejection power of Theorem 1 is not so strong as we can see in simulations in Section 3. We can obtain a more strict test by combining the demand assumption introduced by Carvajal et al. (2014). Even though it is straightforward that the additional assumption in the model provides additional constraints in data, it is less clear that discrete-continuous demand function and the additional assumptions have a non-empty intersection and that we can re-construct a demand system which satisfy both conditions.

In order to define the additional restrictions, I introduce some notations first. Denote $\epsilon_{jt}(p) : R^J_+ \to R$ as the relative decrease in the demand of good j at time t in response to an infinitesimal increase in its price. That is, given the demand function Q_{jt} for good j at time t ,

$$\epsilon_{jt}(p) = -\frac{\partial Q_{j,t}(p_j, p_{-j})}{\partial p_j} \frac{1}{Q_{jt}(p)}$$

Therefore, the own price elasticity is expressed as $p_j \epsilon_{jt}(p)$.

Then we can define the following properties of demand functions.

Definition: A system of demand functions satisfies *co-evolving* property if, for any s and $t \in T$, either

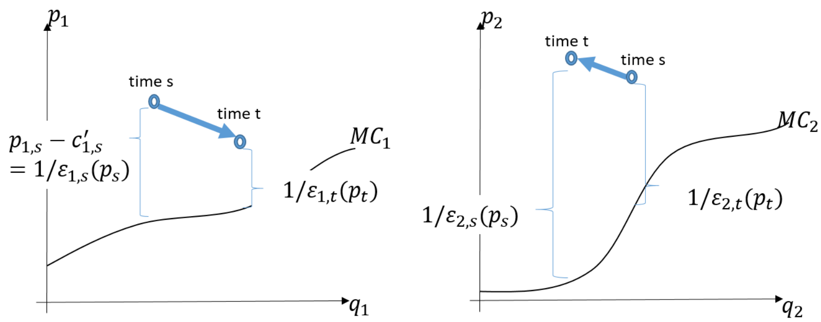
$$\epsilon_{js}(p) \geq \epsilon_{jt}(p) \text{ for all } p \in R_+^J \text{ and all } j \in \mathcal{J}\text{, or} \tag{4}$$

$$\epsilon_{js}(p) \leq \epsilon_{jt}(p) \text{ for all } p \in R_+^J \text{ and all } j \in \mathcal{J} \tag{5}$$

The co-evolving demand property is introduced to capture an idea of *common demand shock* in Carvajal et al. (2013), which is a key component to obtain non-trivial restriction on data in their work. As we can see in the above equations, if a relative slope of demand is higher for firm j in a market t, then so as for other firms $k \neq j$. In other words, we can construct a common order of demands over $\mathcal{T}$ according to the relative slopes for all firms.

The power of the co-evolving property is emphasized in a single product example. Consider the same prices and quantities as the previous example, but two goods are produced by different firms; $(p_{j,\tau}, q_{j,\tau})_{j=1,2,\,\tau=s,t}$ s.t. $\mathcal{J}_1 = \{1\}$, $\mathcal{J}_2 = \{2\}$, $p_{1,s} > p_{1,t}$, $p_{2,s} < p_{2,t}$, $q_{1,s} < q_{1,t}$, and $q_{2,s} > q_{2,t}$ (see Fig. 2). Since two goods are produced by different firms, eq.(1) is no longer satisfied. However, the co-evolving property gives us an alternative restriction. If they are single product firms, the FOC is re-written in the following form; $p_j - C'_{j,t}(Q_{j,t}(p)) = 1/\epsilon_{j,t}(\bar{p}_{j,t})$. Since the marginal costs are increasing, we can obtain the inequality about the profit margins; $1/\epsilon_{1,s}(\bar{p}_s) = \bar{p}_{1,s} - C'_{1,s}(Q_{1,s}(\bar{p}_s)) > \bar{p}_{1,t} - C'_{1,t}(Q_{1,t}(\bar{p}_t)) = 1/\epsilon_{1,t}(\bar{p}_t)$ for firm 1. Similarly, we also have $1/\epsilon_{2,s}(\bar{p}_s) < 1/\epsilon_{2,t}(\bar{p}_t)$. Therefore, the data implies $\epsilon_{1,s}(\bar{p}_s) < \epsilon_{1,t}(\bar{p}_t)$ and $\epsilon_{2,s}(\bar{p}_s) > \epsilon_{2,t}(\bar{p}_t)$. By combining a property that $\epsilon_{j,s}(\cdot)$ is non-decreasing in own price and decreasing in other's price, we have $\epsilon_{1,s}(p) < \epsilon_{1,t}(p)$ but $\epsilon_{1,s}(p) > \epsilon_{1,t}(p)$, which is a contradiction to the co-evolving property. In the following, I refer $\epsilon_{j,t}(\cdot)$ non-decreasing in own price as *log-concave*, and $\epsilon_{j,t}(\cdot)$ decreasing in other's price as *substitutable*, following to Carvajal et al. (2014).

Figure 2: Example: Rejection by Co-evolving Property



Before going to a proposition, I exemplify that the discrete-continuous model and co-evolving property have a non-empty intersection. In the multinomial logit demand, the co-evolving property is satisfied when v_{jt} and v_{kt} move similarly over time. Since the logit demand function (with common M_t over t) requires $\epsilon_{jt}(p) = \alpha - \frac{\alpha}{M} Q_{j,t}(p)$, $\epsilon_{jt}(p) \geq \epsilon_{js}(p)$ holds if and only if $Q_{j,t}(p) \geq Q_{j,s}(p)$ holds. The co-evolving property under the logit demand function requires $Q_{j,t}(p) \geq Q_{j,s}(p)$ if and only if $Q_{k,t}(p) \geq Q_{k,s}(p)$. It can be satisfied when the change of relative value of outside option dominates the change in demand functions. Thus, there is a non-empty intersection of the two property. Log-concavity (of $Q_{j,t}(p)$) is also satisfied if $\frac{h''_j(p_j)}{-h'_j(p_j)}$ is non-decreasing in p_j, and substitutability is always satisfied in discrete-continuous model. In the following proposition, I combine the discrete-choice model and the co-evolving property to derive a set of necessary necessary conditions of data to be rationalized by price competition.

Proposition 1: The set of observations $\{p, q\}$ is Bertrand rationalizable under convex cost function and discrete-continuous demand function with log-concavity, and co-evolving only if there is a permutation of $\mathcal{T}$, denoted by the function $\sigma : \mathcal{T} \to \mathcal{T}$, and real numbers $\alpha_{j,t}$, $\delta_{j,t}$, m_t for all s, $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_j - \delta_{j,t}\} m_t \alpha_{j,t} + \sum_{k \in J_f} \{\bar{p}_k - \delta_{j,t}\} \bar{q}_{k,t}$;
3. $0 \leq \left(\delta_{j,t'} - \delta_{j,t}\right) \left(\bar{q}_{j,t'} - \bar{q}_{j,t}\right)$; and

4. if $\bar{p}_{j,t} \geq \bar{p}_{j,s}$, $\bar{p}_{-j,t} \leq \bar{p}_{-j,s}$ and $\sigma(t) < \sigma(s)$, then $\alpha_{j,k} - m_t^{-1}\bar{q}_{j,t} \leq \alpha_{j,s} - m_s^{-1}\bar{q}_{j,s}$

See Appendix for the proof.

The last condition comes from the co-evolving property and log-concavity, which characterize the common order of $\epsilon_{jt}(p)$ over time. Under the discrete-continuous demand model; $\epsilon_{jt}(\bar{p}) = \frac{h''_{j,t}(\bar{p}_k)}{-h'_{j,t}(\bar{p}_k)} - m_t^{-1}Q_{j,t}(\bar{p}) = \alpha_{j,k} - m_t^{-1}\bar{q}_{j,t}$. The permutation σ is constructed to provide the common order of $\epsilon_{jt}(p)$ (if there exists). In this proposition, I proved only the necessity of the conditions. For the proof of sufficiency, I need to re-construct demand functions satisfying both discrete-continuous structure and co-evolving property from any parameters satisfying the conditions 1-4.

3 Implementation and Simulation

3.1 Implementation

In the tests I introduced in this paper, I need to check the existence of a set of parameters which satisfies all inequalities defined in each test. In contrast to Carvajal et al. (2014), constraints are not linear in parameters. Therefore, we cannot use a technique in linear programming. [5] Therefore, I consider a criterion function similar to that for moment inequalities;

$$Q(\theta; \bar{p}, \bar{q}) = \sum \left((g(\theta; \bar{p}, \bar{q}))_- \right)^2$$

for a vector of inequalities $0 \leq g(\theta; \bar{p}, \bar{q})$. I accept the model if and only if the minimized criteria is (close to) zero. If the data is rationalized by the price competition, deviation of criteria from zero should be only from computational errors since all the conditions should be satisfied for sure (i.e., the criteria can achieves exactly zero) . In the following simulation, I accept the model if and only if the criteria is less than 1^{-10}. Even though common minimization function such as 'fminsearch' or 'fminunc' in MATLAB cannot characterize the whole set of parameters which satisfy inequalities (because the criteria would be flat for such a set of parameters), it is enough for our purpose to find one parameter which satisfies the inequalities if there exists.

[5] For instance, we can just run 'linprog' in MATLAB to check the existence the parameters if constraints are linear in parameters.

3.2 Simulation

In this subsection, I examine performance of the revealed preference tests in simulated data. In the simulation, I first generate a set of prices and quantities ($\bar{p}$, $\bar{q}$) from FOCs under the logit demand functions and quadratic cost functions, given a set of parameters. Next, I perturb the obtained price and quantity. In this exercise, the perturbed data is generated by $U\left(\left(1-a\right)\bar{p}_{j,\,t},\,\left(1+a\right)\bar{p}_{j,\,t}\right)$ and $U\left(\left(1-a\right)\bar{q}_{j,\,t},\,\left(1+a\right)\bar{q}_{j,t}\right)$ for each $j \in \mathcal{J}$ and $t \in \mathcal{T}$, where $0 < a < 1$. The draws are independent between price and quantities and over $j \in \mathcal{J}$ and $t \in \mathcal{T}$. For each specification and each parameter value, I generate such perturbed data 100 times and report the rejection ratio. We can expect that the test rejects more data when we increase a since the generated data deviates from original $\bar{p}$ and $\bar{q}$ more largely. Such a tendency is observed in Table 1. Even though the increase of rejection ratio in a is observed in most specifications, some specifications slightly violate this pattern. It is considered that, in some simulation, the minimization procedure stuck at a local minimum and reject a simulated data falsely. More stable implementation procedures are wanted to implement this test. In this exercise, the rejection ratio can be substantially larger for the simulation of single product competition than that for multi-products case (e.g., comparison between competition of single product by each firm and that of two products by each firm). It still is reasonable since, if firms add products, different true prices and quantities can be generated given the same parameter values.

To capture effects only from increase of the number of products, I also run an exercise where the first, second, and third goods of a specific firm are the same (but they can be different over periods, and different for different firms). In Table 2, it is shown that additional goods with the same value as the existing goods provides more restrictions.

4 Summary

In this paper, I modify a test for Bertrand assumption introduced by Carvajal et al. (2014), and made it implementable for Bertrand competition among multi-products firms. To deal with difficulties caused by cannibalization effects, I employ the discrete-continuous demand function introduced by Nocke and Schutz (2016), which includes the multinomial logit demand function and CES demand function as special cases. In the main theorem, I provide the necessary and sufficient

Table 1: Performance in the Simulated Data

		1 product by each firm				2 products by each firm				3 products by each firm			
	a	0.1	0.3	0.5	0.9	0.1	0.3	0.5	0.9	0.1	0.3	0.5	0.9
2 periods	Discrete-Continuous	0.14	0.36	0.35	0.32	0	0.03	0.05	0.25	0.4	0.62	0.78	0.8
	Logit	0.1	0.28	0.32	0.46	0.02	0.23	0.2	0.4	0.59	0.75	0.83	0.82
3 periods	Discrete-Continuous	0.26	0.64	0.64	0.6	0.12	0.35	0.37	0.57	0.9	0.96	0.96	1
	Logit	0.12	0.51	0.72	0.8	0.66	0.84	0.71	0.69	0.93	0.89	0.9	0.97

Note: In this exercise, two firms are assumed to exists. Each firm produces 1, 2, or 3 products over 2, or 3 periods. As a true model, I used logit demand functions $Q_{jt} = \frac{\exp\left(v_{jt} - \alpha_t p_{jt}\right)}{1 + \sum_k \exp(v_{kt} - \alpha_t p_{kt})}$ and quadratic cost functions $C_j'(q_{jt}) = a_j + b_j q_{jt}$. In this exercise, I reject data when the minimized criteria is more than 1^{-10}. In this exercise, a_j and b_j can be different over products, v_{jt} can be different over products and periods, but they are constant over different random draws.

Table 2: Performance in the Simulated Data

		1 product by each firm				2 products by each firm				3 products by each firm			
	a	0.1	0.3	0.5	0.9	0.1	0.3	0.5	0.9	0.1	0.3	0.5	0.9
2 periods	Discrete-Continuous	0	0.03	0.16	0.21	0.01	0.13	0.26	0.29	0.04	0.31	0.43	0.31
	Logit	0	0.11	0.2	0.28	0.01	0.38	0.44	0.39	0.07	0.65	0.65	0.4
3 periods	Discrete-Continuous	0	0.07	0.31	0.41	0.15	0.3	0.61	0.59	0.43	0.55	0.74	0.69
	Logit	0.01	0.12	0.33	0.55	0.2	0.84	0.86	0.72	0.66	0.97	0.94	0.65

Note: In this exercise, two firms are assumed to exists. Each firm produces 1, 2, or 3 products over 2, or 3 periods. As a true model, I used logit demand functions $Q_{jt} = \frac{\exp\left(v_{jt} - \alpha_t p_{jt}\right)}{1+\sum_k \exp(v_{kt} - \alpha_t p_{kt})}$ and quadratic cost functions $C_j'(q_{jt}) = a_j + b_j q_{jt}$. In this exercise, I reject data when the minimized criteria is more than 1^{-10}. In this exercise, v_{jt} can be different over firms and periods, but same among products produced by the same firm. On the other hand, a_j and b_j are the same for all products. Furthermore, they are constant over different random draws.

condition for data to be rationalized by Bertrand competition among multi-products firms under the discrete-continuous model. The test is implementable without any IVs, and the rejection by the suggested test deterministically implies misspecification of the model rather than peculiar realizations of structural error terms. The simulated data show that the model itself can provide a tight restriction on observed data especially if each firm produces many products. Even though the conditions obtained in the main theorem is necessary and sufficient conditions, an implementing procedure suggested in the paper seems unstable. Elaboration of the implementation procedure is wanted in the future research.

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Appendix

Proof of Theorem 1. For sufficiency, it is enough to construct cost functions and demand functions for each firm which construct a profit function maximized at $\bar{p}_{j,t}$, $\bar{q}_{j,t}$.

First, consider the re-construction of demand function. If data satisfies the restriction defined in Theorem 1, we should be able to find $\alpha_{j,t}$ which corresponds to $\frac{h''_{j,t}(\bar{p}_{j,t})}{-h'_{j,t}(\bar{p}_{j,t})}$ for each observation, where $h_{j,t} : R_+ \to R$ composes the true data generating process. In the reconstruction of demand functions, I consider $\bar{h}_{j,t} : R_+ \to R$ s.t. $\frac{\bar{h}''_{j,t}(p_j)}{-\bar{h}'_{j,t}(p_j)} = \alpha_{j,t}$ for all $p_j \in R_+$. This is an analogous of the construction of utility function in Afriat (1967), where the gradient of utility function is assumed to be locally constant. Since the constant $\frac{\bar{h}''_{j,t}(p_j)}{\bar{h}'_{j,t}(p_j)}$ implies that $\bar{h}_{j,t}(p_j)$ can be represented as CARA function with risk averseness $\alpha_{j,t}$, $\bar{h}_{j,t}(p_j) = \exp\{v_{jt} - \alpha_{jt}p_j\}$ for some v_{jt}. Then, we can construct a demand function, $\bar{Q}_{j,t}(p) = m_t \frac{-\bar{h}'_{j,t}(p_j)}{H_0 + \sum_k \bar{h}_{k,t}(p_j)} = m_t \frac{\alpha_{jt}\exp\{v_{jt}-\alpha_{jt}p_j\}}{H_0+\sum_k \exp\{v_{kt}-\alpha_{kt}p_j\}}$. Here, I denote the reconstructed demand function as $\bar{Q}_{j,t}(p)$ in order to distinguish from the demand function in true data generating process, $Q_{j,t}(p)$. Now, v_{jt} can be chosen to satisfy a system of K equations; $m_t \frac{\alpha_{jt}\exp\{v_{jt}-\alpha_{jt}\bar{p}_{jt}\}}{H_0+\sum_k \exp\{v_{kt}-\alpha_{kt}\bar{p}_{jt}\}} = \bar{q}_{jt}$ for all j, in the same way as the inversion of share function in logit specifications discussed in Berry (1994).

Since (δ_{jt}, q_{jt}) satisfies co-monotone property, we can use monotone cubic interpolation to re-construct increasing and continuously differentiable $\bar{C}'(\cdot)$. Then, we can re-construct $\bar{C}(q) = \int_0^q \bar{C}'(x)\,dx$, which is convex and twice continuously differentiable. [6]

The remaining step is to prove that $(\bar{p}, \bar{q})$ is an equilibrium under reconstructed demand and cost functions. Since the re-constructed profit function is continuously differentiable, FOC must be satisfied at the optimal price. Therefore, it is enough to show that there is the unique solution of FOC for each firm given other firms' strategies. To show that, I use the common ι-markup property examined in Nocke and Schutz (2016). The following part is closely related to the proofs in Nocke and Schutz (2016) (especially, Lemma F), but there are a few differences. First, we don't need to prove the existence of the equilibrium since we already have data as a candidate of equilibrium. Therefore, we just need to show that those data can be an equilibrium. Second, we consider more general cost specification than Nocke and Schutz (2016). It complicates the inversion from μ^f to

[6] Carvajal et al. (2013, 2014) re-construct a cost function as an upper envelop of linear cost functions, whose slope is determined by $\delta_{j,t}$'s. Instead, in this paper, I use cubic interpolation to have the differentiability, which is necessary to invert ι-markup.

price vectors since marginal cost is not a constant, but a function of quantity of the product. Third, the re-constructed demand function is a special case of the demand function in Nocke and Schutz (2016). Therefore, we can circumvent a difficulty from a general cost function by specifying shape of the demand function.

In the following, I omit the subscript for time t since I consider repetition of static NE and the following logic is applied for each t. Then, I denote the reconstructed demand function as $\bar{Q}_j(p) = m\frac{-\bar{h}'_j(p_j)}{H_0+\sum_k \bar{h}_k(p_j)} = m\frac{\alpha_j \exp\{v_j-\alpha_j p_j\}}{H_0+\sum_k \exp\{v_k-\alpha_k p_j\}}$ and $\bar{h}'_j(p_k) = -\alpha_j \exp\{v_j-\alpha_j p_j\}$, $\bar{h}''_j(p_k) = \alpha_j^2 \exp\{v_j-\alpha_j p_j\}$, and $\frac{\bar{h}''_j(p_k)}{-\bar{h}'_j(p_k)} = \alpha_j$. Since we now consider a maximization problem of a specific firm given other firm's strategy, let denote $\bar{h}_0 + \sum_{k\notin J_f} \bar{h}_k(p_k) = H_0$ and $J_f = \{1, ..., n\}$ without loss of generality. By the FOC, we have the following for any j

$$\left\{p_j - \bar{C}'_j\left(\bar{Q}_j(\mathbf{p})\right)\right\}\frac{\bar{h}''_j(p_j)}{-\bar{h}'_j(p_j)} = 1 + m^{-1}\sum_{k\in J_f}\left\{p_k - \bar{C}'_k\left(\bar{Q}_k(\mathbf{p})\right)\right\}\bar{Q}_k(\mathbf{p}) \tag{6}$$

Since RHS is same for any $j \in J_f$, the solution of system of equations defined by (6) for any $j \in J_f$ satisfies

$$\nu_j(\mathbf{p}) \equiv \left\{p_j - \bar{C}'_j\left(\bar{Q}_j(p)\right)\right\}\alpha_j = \mu^f$$

for any $j \in J_f$. Let $\nu(\mathbf{p}) = [\nu_1(\mathbf{p}), ..., \nu_n(\mathbf{p})]'$. Then, $\mathbf{p} = \nu^{-1}\left(\mathbf{1}\mu^f\right) \equiv r\left(\mu^f\right) \equiv \left[r_1\left(\mu^f\right), ..., r_n\left(\mu^f\right)\right]'$ at the solution of (6). Then, we can rewrite the condition (6) as

$$\begin{aligned}
\mu^f &= 1 + m^{-1}\sum_{k\in J_f}\left\{r_k\left(\mu^f\right) - \bar{C}'_k\left(r\left(\mu^f\right)\right)\right\}\bar{Q}_k\left(r\left(\mu^f\right)\right) \\
&= 1 + m^{-1}\sum_{k\in J_f}\underbrace{\left\{r_k\left(\mu^f\right) - \bar{C}'_k\left(r\left(\mu^f\right)\right)\right\}\alpha_k}_{\mu^f}\frac{1}{\alpha_k}\bar{Q}_k\left(r\left(\mu^f\right)\right) \\
&= 1 + m^{-1}\mu^f\sum_{k\in J_f}\frac{1}{\alpha_k}\bar{Q}_k\left(r\left(\mu^f\right)\right) \\
\Leftrightarrow 0 &= 1 + \mu^f\left\{m^{-1}\sum_{k\in J_f}\frac{1}{\alpha_k}\bar{Q}_k\left(r\left(\mu^f\right)\right) - 1\right\} \equiv \psi\left(\mu^f\right)
\end{aligned}$$

Then, the uniqueness of the solution of FOC is proved by strict monotonicity of $\psi\left(\mu^f\right)$. Again, the existence of the solution can be omitted since the data satisfies FOC by the construction of

$\left(\bar{Q}_j\left(\cdot\right), \bar{C}_j\left(\cdot\right)\right)_{j\in J_f}$. By taking a derivative w.r.t. μ^f

$$\psi'\left(\mu^f\right) = \underbrace{\sum_{k\in J_f}\frac{\exp\left\{v_k-\alpha_k r_k\left(\mu^f\right)\right\}}{H_0+\sum_l \exp\left\{v_l-\alpha_l r_l\left(\mu^f\right)\right\}}-1}_{<0} + \underbrace{\mu^f m^{-1}\sum_{k\in J_f}\frac{1}{\alpha_k}\underbrace{\frac{\partial \bar{Q}_k\left(\mathbf{p}\right)}{\partial \mathbf{p}}|_{\mathbf{p}=\bar{\mathbf{p}}}}_{1\times n}\underbrace{r'\left(\mu^f\right)}_{n\times 1}}_{\equiv A}$$

It is enough to show that $A\leq 0$.

$$\begin{aligned} A &= \mu^f m^{-1}\underbrace{\left[\frac{1}{\alpha_1},...,\frac{1}{\alpha_n}\right]}_{1\times n}\underbrace{\frac{\partial \bar{Q}\left(\mathbf{p}\right)}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}}_{n\times n}\underbrace{\frac{\partial \nu^{-1}\left(\mathbf{m}\right)}{\partial \mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f}}_{n\times n}\underbrace{\mathbf{1}}_{n\times 1} \\ &= \mu^f m^{-1}\underbrace{\mathbf{1}'}_{1\times n}\underbrace{\Lambda^{-1}\underbrace{\frac{\partial \bar{Q}\left(\mathbf{p}\right)}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}}_{n\times n}\underbrace{\frac{\partial \nu^{-1}\left(\mathbf{m}\right)}{\partial \mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f}}_{n\times n}}_{\equiv B}\underbrace{\mathbf{1}}_{n\times 1} \end{aligned}$$

where $\Lambda=\begin{bmatrix}\alpha_1 & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n\end{bmatrix}$. Since $\mu^f>0$ and $m>0$, it is enough to show that B is negative semi-definite.

In order to consider derivatives of $\nu^{-1}\left(\mathbf{m}\right)$, we first consider the derivative of ν. Recall that $\nu_j\left(\mathbf{p}\right)\equiv\left\{p_j-C_j'\left(\bar{Q}_j\left(p\right)\right)\right\}\alpha_j$. then, partial derivatives are;

$$\begin{aligned} \frac{\partial \nu_k\left(\mathbf{p}\right)}{\partial p_k} &= \alpha_k\left(1-c''\left(\bar{Q}_k\left(\mathbf{p}\right)\right)\frac{\partial \bar{Q}_k\left(\mathbf{p}\right)}{\partial p_k}\right) \\ \frac{\partial \nu_k\left(\mathbf{p}\right)}{\partial p_j} &= -\alpha_k c''\left(\bar{Q}_k\left(\mathbf{p}\right)\right)\frac{\partial \bar{Q}_k\left(\mathbf{p}\right)}{\partial p_j} \end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial \nu(\mathbf{p})}{\partial \mathbf{p}'} &= \Lambda \left\{ I - \begin{bmatrix} c''\left(\bar{Q}_1(\mathbf{p})\right) \frac{\partial \bar{Q}_1(\mathbf{p})}{\partial p_1} & c''\left(Q_1(\mathbf{p})\right) \frac{\partial Q_1(\mathbf{p})}{\partial p_n} \\ \\ c''\left(Q_n(\mathbf{p})\right) \frac{\partial Q_n(\mathbf{p})}{\partial p_1} & c''\left(Q_n(\mathbf{p})\right) \frac{\partial Q_n(\mathbf{p})}{\partial p_n} \end{bmatrix} \right\} \\
&= \Lambda \left\{ I - \Gamma(\mathbf{p}) \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} \right\}
\end{aligned}$$

where $\Gamma(\mathbf{p}) = \begin{bmatrix} c''(Q_1(\mathbf{p})) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c''(Q_n(\mathbf{p})) \end{bmatrix}$. Then,

$$\begin{aligned}
B &= \Lambda^{-1} \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \frac{\partial r(\mathbf{m})}{\partial \mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f} \\
&= \Lambda^{-1} \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \left[\frac{\partial \nu(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \right]^{-1} \\
&= \Lambda^{-1} \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \left[\Lambda \left\{ I - \Gamma(\mathbf{p}) \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} \right\} \right]^{-1} \\
&= \Lambda^{-1} \left(\left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \right)^{-1} \right)^{-1} \left\{ I - \Gamma(\mathbf{p}) \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} \right\}^{-1} \Lambda^{-1} \\
&= \Lambda^{-1} \left(\left\{ I - \Gamma(\mathbf{p}) \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} \right\} \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \right)^{-1} \right)^{-1} \Lambda^{-1} \\
&= \Lambda^{-1} \left(\left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \right)^{-1} - \Gamma(\mathbf{p}) \right)^{-1} \Lambda^{-1} \\
&= -\Lambda^{-1} \left(\Gamma(\mathbf{p}) - \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \right)^{-1} \right)^{-1} \Lambda^{-1}
\end{aligned}$$

Now, B is negative definite as long as $\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}$ is negative semi definite;

$$\begin{aligned}
\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} &= \begin{bmatrix} -m^{-1}Q_1(p)\{\alpha_1 m - Q_1(p)\} & m^{-1}Q_1(p)Q_2(p) & \cdots & m^{-1}Q_1(p)Q_n(p) \\ m^{-1}Q_1(p)Q_2(p) & -m^{-1}Q_2(p)\{\alpha_2 m - Q_2(p)\} & & m^{-1}Q_2(p)Q_n(p) \\ \vdots & & \ddots & \vdots \\ m^{-1}Q_1(p)Q_n(p) & m^{-1}Q_2(p)Q_n(p) & \cdots & -m^{-1}Q_n(p)\{\alpha_n m - Q_n(p)\} \end{bmatrix} \\
&= m^{-1}\begin{bmatrix} -Q_1(p)\{\alpha_1 m - Q_1(p)\} & Q_1(p)Q_2(p) & \cdots & Q_1(p)Q_n(p) \\ Q_1(p)Q_2(p) & -Q_2(p)\{\alpha_2 m - Q_2(p)\} & & Q_2(p)Q_n(p) \\ \vdots & & \ddots & \vdots \\ Q_1(p)Q_n(p) & Q_2(p)Q_n(p) & \cdots & -Q_n(p)\{\alpha_n m - Q_n(p)\} \end{bmatrix} \\
&= m^{-1}\left\{\begin{bmatrix} Q_1(p)Q_1(p) & \cdots & Q_1(p)Q_n(p) \\ \vdots & \ddots & \vdots \\ Q_1(p)Q_n(p) & \cdots & Q_n(p)Q_n(p) \end{bmatrix} - m\begin{bmatrix} \alpha_1 Q_1(p) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n Q_n(p) \end{bmatrix}\right\}
\end{aligned}$$

Then,

$$\begin{aligned}
x'\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}x &= m^{-1}\left\{x'Q(p)Q(p)'x - mx'\begin{bmatrix} \alpha_1 Q_1(p) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n Q_n(p) \end{bmatrix}x\right\} \\
&= m^{-1}\left\{\left(\sum_i x_i Q_i\right)^2 - m\sum_i x_i^2 \alpha_i Q_i\right\} \\
&= m^{-1}\left(\sum_i x_i Q_i\right)^2 - m^{-1}\sum_i x_i^2 Q_i^2 + m^{-1}\sum_i x_i^2 Q_i^2 - \sum_i x_i^2 \alpha_i Q_i \\
&= -m^{-1}\underbrace{\left\{\sum_i x_i^2 Q_i^2 - \left(\sum_i x_i Q_i\right)^2\right\}}_{>0} - m^{-1}\left\{\sum_i x_i^2 Q_i \underbrace{(m\alpha_i - Q_i)}_{>0}\right\} < 0
\end{aligned}$$

Then, $-\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}$ is positive definite, so as $\left(-\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right)^{-1}$. Therefore, $\Gamma(\mathbf{p}) - \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1}$ is positive definite since $\Gamma(\mathbf{p})$ is a diagonal matrix with positive components. Therefore,

$$\begin{aligned}
x'Bx &= -x'\Lambda^{-1}\left(\Gamma(\mathbf{p}) - \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1}\right)^{-1}\Lambda^{-1}x \\
&= -\left(\left(\Lambda^{-1}\right)'x\right)'\left(\Gamma(\mathbf{p}) - \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1}\right)^{-1}\Lambda^{-1}x \\
&= -\left(\Lambda^{-1}x\right)'\left(\Gamma(\mathbf{p}) - \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1}\right)^{-1}\Lambda^{-1}x \\
&< 0
\end{aligned}$$

Therefore, B is negative definite, which gives $\psi'\left(\mu^f\right) < 0$.

Proof of Proposition 1:

For Proposition 1, we need to derive the last condition as a necessary condition.

By co-evolving property, we can find a permutation such that $\sigma(t) < \sigma(s)$ implies $\epsilon_{j,t}(p) \leq \epsilon_{j,s}(p)$ for all $j \in J$ and for all p. Then, if $\bar{p}_{i,t} \geq \bar{p}_{i,s}$, $\bar{p}_{-i,t} \leq \bar{p}_{-i,s}$ and $\sigma(t) < \sigma(s)$, then $\alpha_{j,t} - m_t^{-1}\bar{q}_{j,t} = \epsilon_{j,t}(\bar{p}_{jt}, \bar{p}_{-jt}) \leq \epsilon_{j,t}(\bar{p}_{js}, \bar{p}_{-jt}) \leq \epsilon_{j,s}(\bar{p}_{js}, \bar{p}_{-js}) \leq \epsilon_{j,s}(\bar{p}_{js}, \bar{p}_{-js}) = \alpha_{j,s} - m_s^{-1}\bar{q}_{j,s}$. Thus, $\alpha_{j,t} - m_t^{-1}\bar{q}_{j,t} \leq \alpha_{j,s} - m_s^{-1}\bar{q}_{j,s}$ □