Maintaining Capital in the Presence of Obsolescence

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Abstract

We examine an owner's problem on how best to maintain a capital asset that is subject to both physical wear and tear and obsolescence. The model developed is a non-linear extension of the Thompson model, an early contribution to the optimal-control literature on deterministic maintenance. The model is estimated from the U.S. data on rent and maintenance expenditures on office buildings, and is then used to carry out counterfactual simulations. It is shown that the path of an optimal maintenance expenditure is different according to the asset being a high-deterioration type or not (*i.e.*, whether it wears out rapidly without maintenance or not). Also, it is demonstrated that a high obsolescence rate and a high deterioration rate could have different effects on an optimal maintenance policy.

1 Introduction

Maintenance and repair are a large part of economic activity. In Canada, expenditures on maintenance and repair averaged 5.7% of the Canadian GDP over the period 1961–1993, which was four times as large as R&D expenditures (1.4%) and only slightly smaller than public spending on education (6.8%); McGratten and Schmitz (1999). In spite of this significance, the economic literature on maintenance is relatively scant. This paper studies the investment decision in capital maintenance in the presence of obsolescence. Namely, how much should be spent on maintenance if the value of a capital asset declines as time passes because of increased wear and tear and obsolescence?

An issue necessarily examined in this problem is a dynamic tradeoff facing an owner. A smaller expenditure on maintenance may raise an owner's receipts in the short run, but it also may diminish an asset's capability because of increased wear and tear, leading to lower

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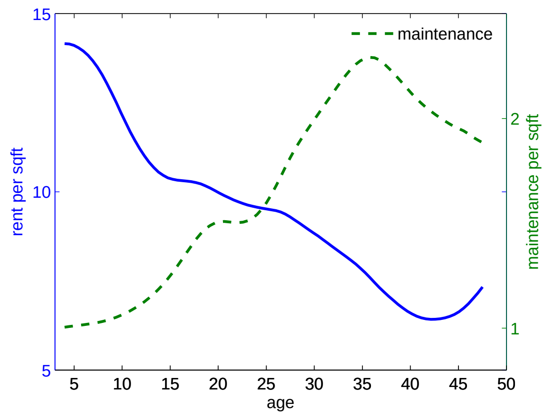


Figure 1: Kernel estimates of "rent" or "rental rate" and maintenance

profits in the future. A maintenance policy should balance these short-term and long-run profits, in some sense, optimally. Generally the incentive for maintenance is larger, the larger is the foregone profit from reduced maintenance (*i.e.*, the opportunity cost of maintenance), while the latter may be sensitive to asset types as well as to current and future market conditions. An owner's investment behavior in maintenance may thereby vary interestingly, depending on these attributes.

Figure 1 displays estimates of rent and maintenance expenditure on an office building as functions of age. Here, the time series of rent is drawn in a solid line, while that of maintenance expenditure is drawn in a dotted line. As can be seen, the rent steadily declines until the building is approximately 43 years old. A possible reason for an increase in rent beyond the age of 43 years is remodeling or refurbishing the building. To the extent that this is the case, it could in essence be considered as a new or different building. The rents at the ages of 4 and 43 years are about $14.0 per square foot and $6.3 per square foot, respectively. The maintenance expenditure increases until the building is approximately 36 years old and then starts falling. The maintenance outlays at the ages of 4 years, 36 years and 43 years are about $1.0 per square foot, $2.3 per square foot and $1.8 per square foot, respectively.

Given these observations, one may ask why the time path for maintenance expenditure first increases and then decreases, rather than first decreases and then increases or keeps increasing, while that of rent steadily declines over time. What asset and market attributes are relevant to this pattern of rent and maintenance expenditure paths?

An optimal control model of capital maintenance is developed to address these issues. An indivisible capital asset is owned to be used for productive purposes for a length of time and then sold. The production revenue depends on physical wear and tear and obsolescence.

Maintenance slows down physical wear and tear but has no effect on obsolescence. The model developed here may be viewed as a nonlinear extension of Thompson's model (1968), an early contribution to the literature, which is reviewed shortly.

It will be shown analytically that an optimal maintenance policy may vary qualitatively with asset type, *i.e.*, whether an asset wears out rapidly without maintenance or not. Specifically, if an asset deteriorates quickly with time and use without maintenance, then an optimal maintenance should decrease. If an asset wears out slowly without maintenance, an increasing maintenance expenditure can be optimal. The results are intuitive, since if the deterioration of an asset without maintenance is slow, then the forgone profit from a smaller initial investment in maintenance may not be large. In that case, a minor scratch may optimally be left being unrepaired, to be maintained in the future. If the asset wears out rapidly without maintenance, then such a policy causes a significant loss in the future profit. This dependence of an optimal maintenance policy on asset type has not been explored fully in the literature and thereby merits emphasis.

The optimal control literature on capital maintenance dates from Näslund (1966). Explicit modeling starts with Thompson (1968) and Kamien and Schwartz (1971). While Thompson proposes a deterministic maintenance model, the model due to Kamien and Schwartz is probabilistic. A key characteristic of deterministic capital maintenance is that a capital asset does not break down during the service life for which a given maintenance policy is established; it can deteriorate but still remains operable at some level. A probabilistic maintenance model considers the possibility of the cessation of production even in the provision of certain maintenance. While Thompson (1968) relates the physical condition of an capital asset to an owner's operating receipts, Kamien and Schwartz (1971) relate it to the probability of breakdown and consider a constant production revenue.

Both studies have been advanced by subsequent researchers, such as Arora and Lele (1970), Virtanen (1982), Mehrez and Berman (1994), Dogramaci and Fraiman (2004) and Bensoussan and Sethi (2007). [1] There is a large body of literature on maintenance and replacement under Markovian deterioration. A recent extensive review is in Wang (2002).

This paper studies a deterministic maintenance problem and adds to the optimal-control literature on deterministic capital maintenance. First, the model developed here is a nonlinear model in maintenance expenditure, a control variable. Being linear in maintenance

[1] Arora and Lele (1970) suggest a minor modification of the Thompson model. Virtanen (1982) incorporates a possibility of breakdown into the Arora and Lele model. Mehrez and Berman (1994), based also on the Arora and Lele model, introduce a hazard rate and the opportunity of a chain of replacement, where the hazard rate is about technological breakthrough. Dogramaci and Fraiman (2004), on the other hand, include the possibility of a chain of replacement in the model of Kamien and Schwartz. Bensoussan and Sethi (2007) reformulate Kaimien and Schwartz's original problem using dynamic programming technique.

expenditure, the previous models in the literature, including Thompson (1968), seek an optimal maintenance policy that is bang-bang and exclusively focus on the switching time (*i.e.*, the timing to start and stop maintenance efforts). In contrast, our model is a non-linear extension of the Thompson's model (1968). This enables us to explore interesting properties on the evolution of an optimal maintenance expenditure policy.

Second, we distinguish between maintenance and partial replacement. In the previous models in the literature, maintenance is assumed to be effective at preventing both physical wear and tear and obsolescence. However, it may be called partial replacement rather than maintenance that counteracts the impact of obsolescence. Assuming that maintenance is effective at preventing physical wear and tear but has nothing to do with the impact of obsolescence, we make this distinction explicit. A simulation will demonstrate that an increase in physical wear and tear may have a different effect on maintenance expenditure than that of obsolescence. Although both reduce an owner's receipts, an increase in physical wear and tear can optimally induce more maintenance, whereas an increase in obsolescence always reduces optimal maintenance incentives. An optimal maintenance policy may vary if partial replacement is considered to be a part of maintenance efforts or not, so the distinction matters.

The rest of the paper is organized as follows. In Section 2, the model is explained. In Section 3, an optimal policy is characterized. In Section 4, the parameters of the model are estimated from the data on rent and maintenance expenditure of an office building displayed in Figure 1. Section 5 concludes.

2 The Model

Time is continuous and is indexed by $t \in [0, Z]$ with $0 < Z < \infty$. An indivisible capital asset is owned at $t = 0$, used for productive purposes for a length of time and then sold by $t = Z$. Let $T \in (0, Z]$ be a sale date. An owner receives a flow of nonnegative production revenue over $(0, T)$ and a lump-sum resale profit at $t = T$. These receipts are larger, the more relatively capable is the asset at the moment. Specifically suppose that the relative capability of the asset at time t is given by

$$\bar{c}(t) - c(t) = [\bar{c}(0) - c(t)] + [\bar{c}(t) - \bar{c}(0)] \equiv a(t) + b(t)$$

where $\bar{c}(t)$ is the capability of an asset that embodies the *best* technology available at time t, $c(t)$ is the capability of the owner's asset at time t, $a(t)$ is the state of deterioration caused by increased wear and tear and $b(t)$ is the state of obsolescence due to technical advances. Let $R(a(t) + b(t))$ and $S(a(t) + b(t))$ be the production revenue and the resale price respectively. Assume that both decrease more than proportionally as $a(t) + b(t)$ increases. That is,

$$R'(\cdot) < 0, \qquad R''(\cdot) < 0, \qquad S'(\cdot) < 0, \qquad S''(\cdot) < 0.$$

The owner invests in maintenance at a constant marginal and average cost that is unity. Let $m(t)$ be the maintenance expenditure at time t. Assume that maintenance is effective at preventing or decreasing the process of physical wear and tear at a diminishing rate but has no effect on the impact of obsolescence. Specifically, $a(t)$ and $b(t)$ follow the transition functions

$$\begin{aligned} &\dot{a}(t) = \alpha a(t) - z(m(t)), \quad && a(0) = a_0, \quad a(T) = a_T, \quad a(t) \geq a_0 \\ &\dot{b}(t) = \beta, && b(0) = 0 \end{aligned} \tag{1}$$

for some positive constants α and β and the maintenance production function z that is increasing and concave and vanishes when $m(t) = 0$:

$$z'(m(t)) > 0, \qquad z''(m(t)) < 0, \qquad z(0) = 0.$$

In words, without maintenance the state of deterioration $a(t)$ grows at the constant rate α for all $t \in (0, Z)$, and the state of obsolescence $b(t)$ evolves at the constant rate β for all $t \in (0, Z)$. A large α is associated with fast deterioration in the absence of maintenance. So, the larger is α, the faster the asset wears out without maintenance. A large a_0 indicates that the asset is low-tech *relative to those of the same vintage*, since $a(0) = \overline{c}(0) - c(0)$. A large β implies fast technological progress. So, the higher is β, the faster an old asset becomes obsolete. In order to study a non-trivial maintenance problem, suppose $a_0 > 0$. The condition $b(0) = 0$ is by construction. For ease of notation, let $z'(0) \equiv \lim_{m \to 0} z'(m)$. Likewise let $z''(0) \equiv \lim_{m \to 0} z''(m)$.

Assume that the owner has a constant discount rate $r \in (0, 1)$. For a given T, the owner's discounted profit at time 0 is

$$J \equiv \int_0^T e^{-rt} \left[R(a(t) + b(t)) - m(t) \right] \mathrm{d}t + e^{-rT} S(a_T + b(T)). \tag{2}$$

The owner's problem is to choose a sale date $T \in (0, Z]$ and a maintenance expenditure policy $m(t) \in [0, \infty)$ for $0 < t \leq T$ as well as $a_T \in [a_0, \infty)$, so as to maximize J in (2) subject to (1). This is an inequality constrained terminal-date, free terminal-point and salvage value control problem with a control variable $m(t)$, a state variable $a(t)$ and an inequality state constraint $a(t) \geq a_0$. A solution of this program is an optimal policy that will be denoted by $\{T^*, m^*, a_T^*\}$. It is assumed that an optimal policy exists.

A few notations are added regarding the horizon $(0, T)$. An open subinterval of $(0, T)$ will be denoted by I if $a(t) > a_0$ for all $t \in I$. An open subinterval of $(0, T)$ will be denoted by B if $a(t) = a_0$ for all $t \in B$.

In what follows, we suppress the time argument t in the variables m, a and b for ease of notation. For instance, m is meant to be evaluated at time t unless otherwise noted.

3 An Optimal Policy

This section uses the Maximum Principle to describe an optimal policy $\{T^*, m^*, a_T^*\}$. With the costate function $\mu(t)$, form the current-valued Hamiltonian H as

$$H = H(a, m, \mu) = R(a + b) - m + \mu\left[\alpha a - z(m)\right].$$

Then, the Maximum Principle says that if $\{T^*, m^*, a_T^*\}$ maximizes J in (2) subject to (1), then at each instant t it maximizes H subject to (1) as well.

Proposition 1. *Suppose that $\{T^*, m^*, a_T^*\}$ exists. Then, necessarily,*

(i) At an optimal sale date T^*,

$$R(a_T^* + b) - m^* \geq rS(a_T^* + b) - S'(a_T^* + b)(\alpha a_T^* - z(m^*) + \beta) \tag{3}$$

with equality when $T^* < Z$.

(ii) An optimal maintenance policy m^* for $t \in (0, T^*)$ satisfies

$$\begin{cases} 1 = -\mu z'(m^*), & t \in I \text{ and } 1 < -\mu z'(0) \\ m^* = 0, & t \in I \text{ and } 1 \geq -\mu z'(0) \\ m^* = z^{-1}(\alpha a_0), & t \in B. \end{cases} \tag{4}$$

Here μ satisfies the differential equation

$$\dot{\mu} = \begin{cases} (r - \alpha)\mu - R'(a^* + b), & t \in I \\ 0, & t \in B \end{cases} \tag{5}$$

with the terminal condition

$$\mu = S'(a_T^* + b), \qquad t = T^*. \tag{6}$$

The first part of Proposition 1 describes an optimal sale date T^*. The left-hand side (LHS) of (3) is the marginal benefit to the owner from postponing the sale by an instant. By a momentary delay, the owner earns the net profit $R(a_T^* + b) - m^*$. However, at the same time he must give up the interest of the salvage value $rS(a_T^* + b)$. Also the resale price declines by $-\dot{S} = S'(\dot{a}^* + \dot{b})$. These are the marginal costs of postponing the sale. Together (3) states that an optimal sale date should equalize as nearly as possible one to the other.

Part (ii) of the proposition characterizes an optimal maintenance expenditure m^* for $t \in (0, T^*)$ and an optimal endpoint a_T^*. An interpretation is as follows. Begin with (4). Suppose $t \in I$. On the right-hand side (RHS), μ is the current marginal value to the owner of the deterioration rate a^* at time t. Recalling that a is a bad, μ may be thought of as the maximum discounted foregone profit from a unit increase in the deterioration rate a^* at time t (in current value). Adding an additional dollar expenditure on maintenance to m^* at time t prevents a from increasing by $z'(m^*)$. So, the product $-\mu z'(m^*)$ is the marginal benefit to the owner from an additional dollar expenditure on maintenance at time t. Since the marginal cost of maintenance is one, (4) indicates that an optimal maintenance policy m^* for $t \in I$ should equalize as nearly as possible the marginal benefit of maintenance to its marginal cost at each point in time. If $t \in B$, then (4) simply says that the owner's investment in maintenance should be just as much as he needs to keep $a^* = a_0$ or $\dot{a}^* = 0$ at the moment t.

Look at condition (5). Suppose $t \in I$. On the LHS, $\dot{\mu}$ is the rate of change in the marginal value of the deterioration rate a^* at time t. On the RHS, $R'(a^* + b)$ is the rate of decline in the production revenue from a unit increase in the deterioration rate a^* at time t. To consider $r - \alpha$, suppose that a capital asset suffers from a small scratch today and is about to wear out without maintenance. Then, the marginal value μ of the deterioration rate a^*, which is negative, grows at the rate r if this small scratch is left unrepaired, while the foregone profit $-\mu$ from doing so grows at the rate α. Then, the term $r - \alpha$ may be viewed as the *effective* discount rate, and the product $(r - \alpha)\mu$ may be considered as the effective interest of the marginal value of the deterioration rate a^* at time t. Together, (5) says that the change in the marginal value of the deterioration rate a^* at time t should be the sum of the effective interest of the deterioration rate a^* at time t and the marginal decline in the production revenue from a unit increase in the deterioration rate a^* at time t. Alternatively, if $t \in B$, then (5) says that the marginal value of the deterioration rate a^* should be time-invariant.

Turn to condition (6). $S'(a_T^* + b(T^*))$ is the rate of decline in the resale price from a unit increase in the deterioration rate a^* at time T^*. So, (6) states that the marginal value of the deterioration rate a^* at time T^* should be the marginal decline in the resale price from a unit increase in the deterioration rate a^* at the moment.

Lastly, the assertion that μ is the marginal value of the deterioration rate a^* at time t

may be confirmed as follows. Assume $I = (0, T^*)$ and solve (5) with (6):

$$\mu(t) = e^{(r-\alpha)t}\left[e^{-(r-\alpha)T^*}S'(a_T^* + b(T^*)) + \int_t^{T^*} e^{-(r-\alpha)s}R'(a^*(s) + b(s))\mathrm{d}s\right]. \tag{7}$$

As mentioned above, $S'(a_T^* + b(T^*))$ is the marginal decline in the resale price from a unit increase in the deterioration rate a^* at time T^* and $R'(a^*(s) + b(s))$ is the marginal reduction in the production revenue from a unit increase in the deterioration rate a^* at time s. These are the losses incurred by the owner at each moment in time over $[t, T^*]$, if a^* increases by one unit over $[t, T^*]$. All of these are discounted to time t based on the effective discount rate $r - \alpha$. Adding up over $[t, T^*]$, the RHS is the cumulative discounted loss (or the opportunity cost) incurred by the owner from a unit increase in a^* over $[t, T^*]$. So, it seems natural that μ is taken to be the marginal value of the deterioration rate a^* at time t.

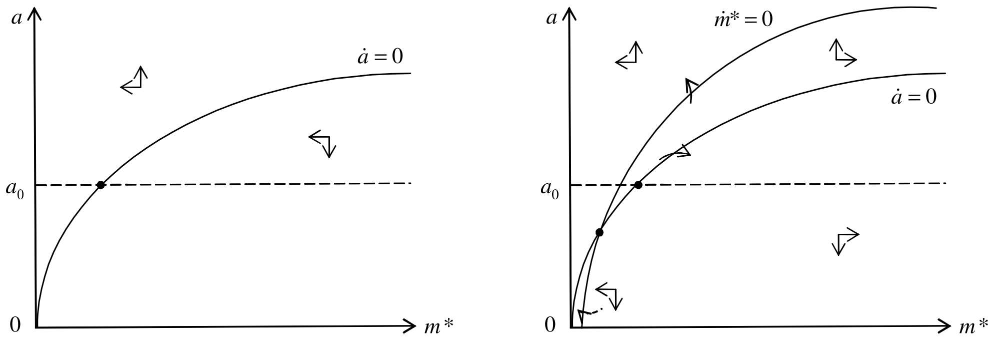
Proposition 1 shows the properties that an optimal sale date T^* and maintenance policy $\{m^*, a_T^*\}$ necessarily possesses. The next proposition establishes a sufficiency result. That is, for a given T^*, if a policy satisfies (4)–(6) then it is optimal.

Proposition 2. *Given T^*, suppose that an optimal policy exists. Given T^*, suppose that $\{m^*, a_T^*\}$ is a policy satisfying (4)–(6). Then, $\{m^*, a_T^*\}$ is optimal.*

Any maintenance policy that satisfies the foregoing necessary conditions (4)–(6) in Proposition 1 is in fact optimal. Hence, the necessary conditions are also sufficient.

Given an optimal sale T^*, the rest of this section discusses the qualitative properties of an optimal maintenance expenditure m^* and its associated deterioration rate a^* for $t \in (0, T^*)$. We restrict our attention to instances where $m^* > 0$ and $a^* > a_0$ for all t.

Suppose that a capital asset suffers from a small scratch and is about to wear out without maintenance. Then, the owner may be indifferent between this small scratch being maintained today and it being left unrepaired today and maintained tomorrow if $r = \alpha$, since the value of the former grows at the same rate as the (negative) value of the latter. Apparently, the necessity of maintenance is larger, the higher is the rate at which the foregone profit from less immediate maintenance increases through time; *i.e.*, for a larger α. So, it can be optimal to leave a small scratch unrepaired today to be fixed in the future if α is small relative to r, but it may not be optimal otherwise. Thus, capital assets may be divided into two types. An asset is said to be of a high deterioration type if $r < \alpha$, and of a low deterioration type if $r > \alpha$. The next propositions reveal the qualitative properties of the paths for an optimal maintenance expenditure m^* and its associated deterioration



(a) $r < \alpha$ (b) $r > \alpha$

Figure 2: Phase diagrams

rate a^* over $(0, T^*)$ for each of these asset types in turn, based on the phase analysis on $(\dot{m}^*, \dot{a}^*)$.

Lemma 1. *In the (m^*, a^*)-space;*

(i) The $\dot{a}^*$=0-isocline exists and is upward sloping, through $(\lim_{m^* \to 0^+} m^*, \lim_{a^* \to 0^+} a^*)$ and is time-invariant.

(ii) The $\dot{m}^*$=0-isocline exists if $r > \alpha$, is upward sloping and shifts down as time passes.

Lemma 1 establishes the basic properties of the isoclines of $\dot{a}^*$=0 and $\dot{m}^*$=0 in the (m^*, a^*)-space. It says that the $\dot{a}^*$=0-isocline exists irrespective of the value of $r - \alpha$, and is upward sloping, through the origin (roughly speaking), and not changing through time. In contrast, the $\dot{m}^*$=0-isocline exists if $r > \alpha$, is upward sloping and shifts down through time.

Figure 2(a) draws the $\dot{a}^*$=0-isocline in the (m^*, a^*)-space when $r < \alpha$, and Figure 2(b) draws the $\dot{a}^*$=0- and the $\dot{m}^*$=0-isoclines in that space when $r > \alpha$. Since the $\dot{m}^*$=0-isocline is time-variant, the two isoclines in Figure 2(b) may be thought of as a snap shot at some time $t \in (0, T^*)$.

Proposition 3. *Let $r < \alpha$. Then, $m^* < z^{-1}(\alpha a_0)$, $\dot{m}^* < 0$, $\dot{a}^* > 0$, and $\ddot{a}^* > 0$ for all $t \in (0, T^*)$*

Proposition 3 provides the qualitative properties of the paths for an optimal maintenance expenditure m^* and its associated deterioration rate a^* for $t \in (0, T^*)$, when an asset concerned is of a high deterioration type. It says that the maintenance expenditure reaches the maximum at the initial date, which is less than $z^{-1}(\alpha a_0)$, and steadily and strictly decreases with time in an optimal plan. Also, the associated deterioration rate takes the minimum value at the initial date and steadily and strictly increases with time at an increasing rate.

Proposition 3 is based on the phase analysis on $(\dot{m}^*, \dot{a}^*)$. Figure 2(a) is the phase diagram in the (m^*, a^*)-space when $r < \alpha$. Each of the horizontal arrows above and below the $\dot{a}^*$=0-isocline indicates the direction of movement of an optimal maintenance expenditure m^*. Likewise, each of the vertical arrows indicates the direction of movement of its associated deterioration rate a^*. The horizontal break line is $a^* = a_0$. This intersects the $\dot{a}^*$=0-isocline at $(m^*, a^*) = (z^{-1}(\alpha a_0), a_0)$. The initial maintenance outlay must be less $z^{-1}(\alpha a_0)$; otherwise, $a^* = a_0$ at the next moment. Since $(m^*(0), a^*(0))$ is above the $\dot{a}^*$=0-isocline, maintenance expenditure decreases through time and the deterioration rate increases though time at time 0 and thereafter in an optimal plan.

Proposition 4. *Let $r > \alpha$. Then, m^* either first increases and then decreases, or evolves monotonically. Moreover, if $\dot{m}^* \leq 0$ at some $t' \in (0, T^*)$, then $\dot{m}^* < 0$ for all $t \in (t', T^*)$.*

Proposition 4 states the qualitative properties of the path of the optimal maintenance expenditure m^* for $t \in (0, T^*)$, when an asset is of a low deterioration type. It says that the evolution of an optimal maintenance expenditure is one of the following three patterns: (i) it increases with time till some time $t' \in (0, T^*)$ and steadily declines with time beyond time t'; (ii) it keeps increasing with time; or (iii) it keeps falling with time. Rephrased, it rules out time paths that are cyclical or are U-shaped (*i.e.*, first decrease and then increase).

Figure 2(b) is the phase diagram of $(\dot{m}^*, \dot{a}^*)$ in the (m^*, a^*)-space when $r > \alpha$. Recalling that the $\dot{m}^*$=0-isocline is time-variant, this diagram should be thought of as a snap shot at some time $t \in (0, T^*)$. Moreover, since the $\dot{m}^*$=0-isocline shifts down as time passes, the value of the a^*-intercept of the $\dot{m}^*$=0-isocline is the largest at the initial date and decreases as time passes. As before, the arrows indicate the directions of movement of an optimal maintenance expenditure m^* and its associated deterioration rate a^*. Also, the horizontal break line is $a^* = a_0$.

In order to track the content of this proposition in Figure 2(b), there are three cases to consider. First, suppose that the $\dot{m}^*$=0-isocline has an initial a^*-intercept, ι_0, that is greater than a_0. The initial maintenance outlay is less than $z^{-1}(\alpha a_0)$; otherwise, $a^* = a_0$ at the next moment. So, $\{m^*(0), a^*(0)\}$ is below the $\dot{m}^*$=0-isocline and above the $\dot{a}^*$=0-isocline, implying that both m^* and a^* are increasing with time early on. If ι_0 is *not* far greater than a_0, then the trajectory of (m^*, a^*), which may be heading to the upper right, may encounter the $\dot{m}^*$=0-isocline, which is shifting down, at some time $t' \in (0, T^*)$. If so, then the value of m^* is the largest at time t' and starts declining beyond time t' while a^* keeps increasing. Alternatively, if ι_0 is far greater than a_0, then the trajectory of (m^*, a^*)

may not reach the $\dot{m}^*$=0-isocline before time T^*. So, in this case, both m^* and a^* increase monotonically over $(0, T^*)$. There is yet another case. Suppose that the $\dot{m}^*$=0-isocline has an initial m^*-intercept that is greater than $z^{-1}(\alpha a_0)$. In this case, clearly $\{m^*(0), a^*(0)\}$ is above both of the $\dot{m}^*$=0- and the $\dot{a}^*$=0-isoclines. Since the $\dot{m}^*$=0-isocline shifts down as time passes, this implies that m^* decreases with time while a^* increases with time over $(0, T^*)$.

Propositions 3 and 4 show that the time path of an optimal maintenance expenditure m^* may vary qualitatively according to the asset's type. Specifically, it is shown that if an asset is of a high deterioration type (that is, an asset wears out quickly without maintenance), an optimal maintenance is always decreasing. If an asset is of a low deterioration type (that is, an asset wears out slowly without maintenance), an increasing maintenance expenditure may be optimal. The results are intuitive since, if an asset deteriorates slowly even without maintenance, then the forgone profit from spending fewer initial dollars on maintenance may not be large. In that case, a minor scratch may optimally be left being unrepaired, to be maintained later. If the asset wears out rapidly, such a policy causes a significant loss in the future profit due to possibly large accumulated physical wear and tear. This dependence of an optimal maintenance policy on asset type has not hitherto been explored fully in the literature and hence merits emphasis.

We conclude this section with the following comparative dynamics results. As before, we continue to restrict our attention to instances where $m^* > 0$ and $a^* > a_0$ for all $t \in (0, T^*)$, given T^*.

Proposition 5. *$\partial m/\partial\beta > 0$ for all $t \in (0, T^*]$ does not occur in an optimal plan.*

Proposition 5 is the comparative dynamics result on the effect of an increase in the obsolescence constant β on the path of an optimal maintenance expenditure m^* for $t \in (0, T^*)$ where $m^* > 0$ and $a^* > a_0$ for all t. It says that an increase in β does *not* raise the maintenance investment for *all* $t \in (0, T^*)$ in an optimal plan. So, increased obsolescence should reduce maintenance efforts at some $t \in (0, T^*)$.

Given T^*, let V be the owner's maximized net discounted production profit; that is,

$$V(\alpha, \beta) \equiv \int_0^{T^*} e^{-rt} \left[R(a^* + b) - m^*\right] \mathrm{d}t.$$

The following proposition establishes how V changes with respect to the decay and obsolescence constants α and β.

Proposition 6. *$V_\alpha(\alpha, \beta) < 0$, $V_\beta(\alpha, \beta) < 0$ and $V_{\beta\beta}(\alpha, \beta) > 0$.*

	mean	std. dev.	min	max
size (square feet)	$254,670$	$304,530$	$10,656.6$	$2,860,100$
maintenance/sq. ft. (1997)	1.5936	0.97363	0	6.3168
rent/sq. ft. (1997)	10.352	5.316	0.06557	43.432
age	26.875	22.614	2	144

Table 1: Summary statistics

Proposition 6 characterizes the properties of the maximized net discounted production profit V. It says that the maximized net discounted production profit V decreases with the decay constant α, and also that V decreases with the obsolescence constant β at a diminishing rate. So, V is convex in β.

4 Estimation

This section analyzes U.S. data on rent and maintenance expenditures on office buildings in the context of the model set out above.

4.1 Data

The office building data were collected by Gort, Greenwood and Rupert (1999) and were originally assembled from the Building Owners and Managers Association International (BOMA). The data set consists of two panels: a five-year panel and a nine-year panel. The five-year panel covers approximately 200 office buildings across the United States from 1989 to 1997. The nine-year panel covers approximately 800 office buildings across the United States from 1993 to 1997. Both include information on age, size, rent and several categories of expenses.

Table 1 displays some summary statistics for the sample. The average size building is $254,670$ square feet, the size of the smallest is $1,0656.6$ square feet. and the size of the largest is $2,860,100$ square feet. The oldest building is 144 years old and the newest is 2 years old. (In each panel, each building was in every year of the sample.)

Figure 1 displays plots of the Kernel estimates of rent and maintenance expenditure on an office building as functions of age. Here, we have chosen a Gaussian (normal) kernel function, one of the most common choice of kernel function, with the bandwidth that is optimal in the sense of minimizing the MISE (mean integrated squared error) in an asymptotic sense.[2] As can be seen, the rent declines over time until the building is approximately

[2] An alternative common choice of kernel function is a bisquare function. We also tried this function. The results were similar.

43 years old. A possible reason for an increase in rent beyond the age of 43 years is remodeling or refurbishing the building. To the extent that this is the case, it could in essence be considered as a new or different building. The rents at the ages of 4 years and 43 years are approximately \$14.0 per square foot and \$6.3 per square foot respectively. The maintenance expenditure increases until the building is about 36 years old and then starts falling. The maintenance outlays at the ages of 4 years, 36 years and 43 years are approximately \$1 per square foot, \$2.3 per square foot and \$1.8 per square foot respectively.

4.2 Parameterization

The maintenance production function has the form

$$z(m) = \zeta \ln(m+1),$$

so that $z'(m) = \zeta/(m+1) > 0$ and $z''(m) = -\zeta/(m+1)^2 < 0$ as specified in Section 2. The production revenue function has the form

$$R(a+b) = \max\{0, \rho_0 + \rho_1 \ln(\rho_2 - a - b)\},$$

so that $R'(a+b) = -\rho_1/(\rho_2 - a - b) < 0$ and $R''(a+b) = -\rho_1/(\rho_2 - a - b)^2 < 0$ as long as $\rho_1 > 0$ and $\rho - a - b > 0$. The resale price function has the form

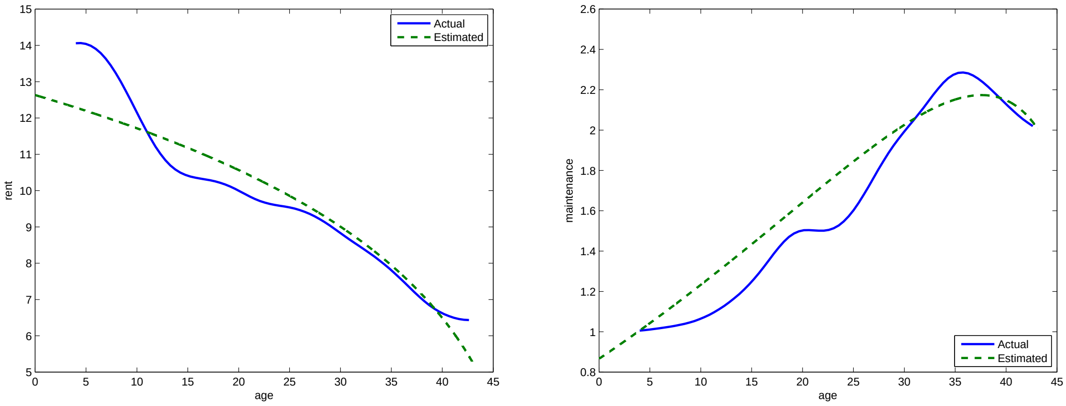
$$S(a+b) = \sigma_0 + \sigma_1 R(a+b),$$

so that $S'(a+b) = \sigma_1 R'(a+b) < 0$ and $S''(a+b) = \sigma_1 R''(a+b)$ as long as $R'(a+b) < 0$ and $R''(a+b) < 0$. Here, the resale price is simply a fraction of the production revenue plus some constant.

With this parameterization, the model's parameters are r, a_0, α, β, ρ_0, ρ_1, ρ_2, σ_0, σ_1 and ζ. These values are estimated as follows. Fix a sale date T^*. Given parameter values of r, α, β, a_0, ρ_0, ρ_1, ρ_2 and ζ, conditions (4) and (5) and the transition function (1) together with a guess on the initial maintenance expenditure $m^*(0)$ imply time series of the deterioration rate a^*, the obsolescence rate b, rent $R(a^* + b)$ and maintenance expenditure m^* as well as the costate variable μ for $t = (0, T^*]$ *with the left-continuity of* m^* *at the sale date* T^* *assumed.* A set of the parameter values and the initial maintenance expenditure is chosen so that the time paths for rent and maintenance expenditure generated by the model fit closely to the data. Applying the estimates of a^*, b and μ at time T^*, condition (6) implies the value of parameter σ_1. With this, condition (3) implies the value of parameter σ_0.

r	a_0	α	β	ρ_0	ρ_1	ρ_2	σ_0	σ_1	ζ
0.08	1.073	0.045	0.115	4.5	4.25	7.85	-149.85	16.71	0.05

Table 2: The parameter estimates



(a) rent (b) maintenance

Figure 3: Actual and estimated time paths

4.3 Parameter Values

Parameter values are presented in Table 2. Figure 3 repeats the plots of Figure 1 and superimposes the model's predicted time paths for rent and maintenance expenditure *with an assumed optimal sale date of* 43 *years*. Figure 3 shows that the model succeeds in tracking the monotonic decline in rent and the non-monotonic time path for maintenance expenditure that first increases and then decreases. The model underestimates the rent and its declining rate for the first ten years but accurately tracks the time path for the 15 years from the age of 25 years to the age of 40 years. Although the model overestimates maintenance expenditure for the 25 years from the age of 5 years to the age of 30 years and underestimates its increasing rate, it predicts that maintenance expenditure reaches the maximum at the age of 37.2 years. In sum, we believe that the model does a reasonable job of predicting the data and that the actual and predicted time paths for rent and maintenance expenditure are both close to each other.

The two-sample Kolmogorov-Smirnov test (KS test) is one of the most useful and general nonparametric methods of comparing the "closeness" of two samples. We have conducted the KS test and not been able to reject the null hypothesis that the two datasets (actual and predicted) are from the same distribution at the 1% significance level. Table 3 is the result of the KS test and includes the KS statistics and the p-values for each of the rent and maintenance expenditure data sets.

	KS-statistics	p-value
rent	0.1625	0.22014
maintenance	0.1625	0.22014

Table 3: KS test (no. observation = 90)

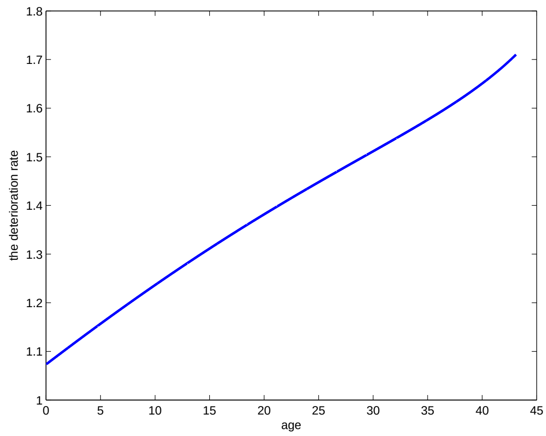


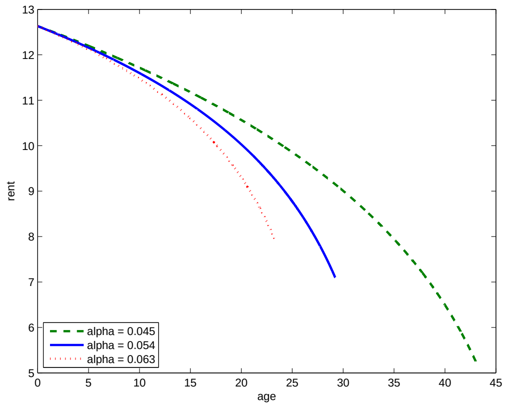
Figure 4: Estimated time path for the deterioration rate

We conclude this subsection with Figure 4, which is the plot of the model's predicted deterioration rate. We see that even with an increasing maintenance expenditure for the first 35 years, the deterioration rate monotonically increases from 1.073 to 1.71 at an almost constant annual rate.

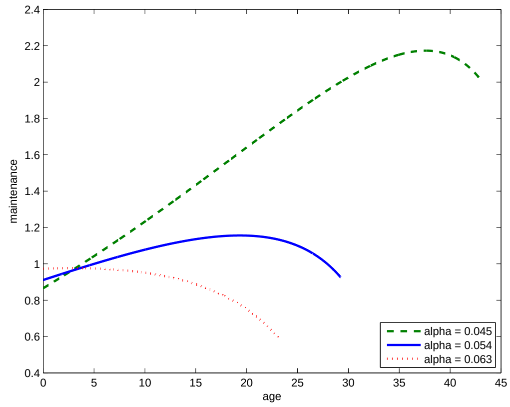
4.4 Counterfactual Simulation

In this section, we use the estimated model to predict the time paths for the optimal maintenance expenditure and its associated rent and deterioration rates for various values of the decay parameter α, the obsolescence parameter β and the initial condition a_0.

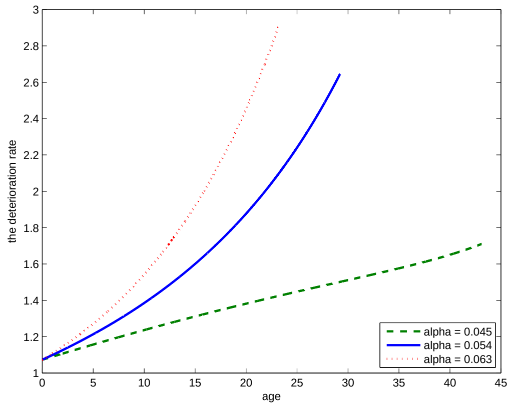
First, the model is simulated for various values of the decay parameter α. A large α implies a fast deterioration of a capital asset without maintenance. Figure 5 is the plot of the time paths of rent, optimal maintenance expenditure and the deterioration rate generated by the model when $\alpha = 0.045$ (estimated from the data), when $\alpha = 0.054$ (raised by 20%) and when $\alpha = 0.063$ (raised by 40%). In all cases, $r < \alpha$, so the asset is a low deterioration type. First, we see that the optimal sale date is approximately 43 years, 30 years and 24 years when $\alpha = 0.045$, 0.054 and 0.063 respectively. Hence, the optimal lifetime is shortened by an increased α. Also, we see that the time series of the optimal maintenance expenditure is very much different for each α. First, when $\alpha = 0.054$, an opti-



(a) rent



(b) maintenance

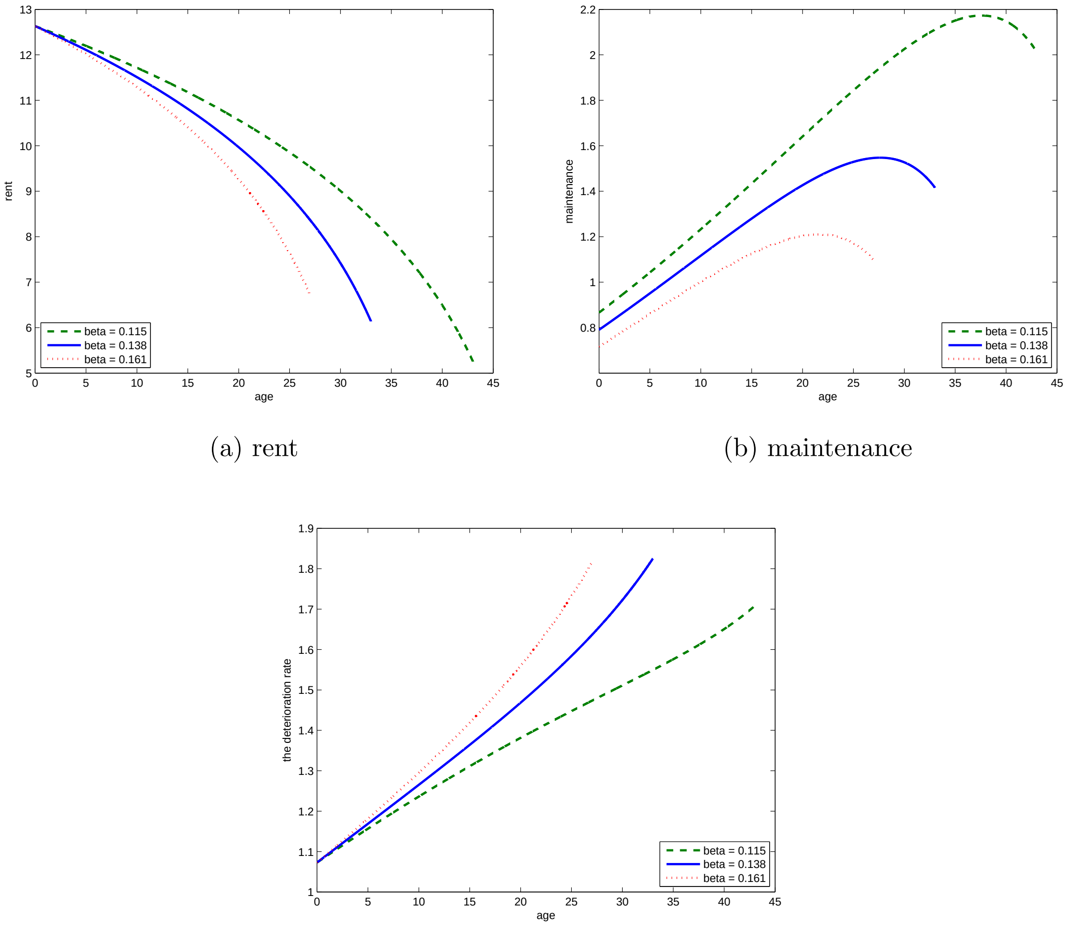


(c) the deterioration rate

Figure 5: Estimated time paths for various α

mal maintenance expenditure still follows the increase-decrease pattern but is flatter than when $\alpha = 0.045$. Second, when $\alpha = 0.063$, the path for an optimal maintenance expenditure is almost non-increasing. The reason that the time path for an optimal maintenance expenditure changes in this fashion as α increases is that an increase in α induces more maintenance effort early on but reduces the maintenance incentives later on. In spite of larger investments in maintenance for the first 10–15 years, the deterioration rate is much higher for increased α and increases at an increasing rate. Consequently, the rent is lower, the larger is α.

Figure 6 simulates and plots the time paths for rent, optimal maintenance expenditure and the deterioration rate for various values of the obsolescence parameter β; that is, when $\beta = 0.115$ (estimated from the data), when $\beta = 0.138$ (raised by 20%) and when $\beta = 0.161$ (raised by 40%). A large β is the indication of fast technological progress, so the



(a) rent

(b) maintenance

(c) the deterioration rate

Figure 6: Estimated time paths for various β

higher is β, the faster an old asset becomes obsolete. As β increases, we see that the asset is sold sooner. Approximately, the sale date is 43 years, 35 years and 27 years when $\beta = 0.115$, 0.138 and 0.161 respectively. So, an optimal lifetime is shorter, the large is β. An increase in β reduces the optimal maintenance expenditure at every point in time, and thus increases the deterioration rate and lowers the rent for all $t \in (0, T^*)$.

Figures 5 and 6 show that an increase in α can have a different effect on optimal maintenance expenditure than does an increase in β. Although both reduce the rent, increased α can encourage maintenance efforts, especially early on, while increased β lowers maintenance incentives all the time. This is because maintenance is effective at preventing or decreasing physical wear and tear but has nothing to do with the effect of obsolescence. It is partial replacement, not maintenance, that counteracts the effects of obsolescence. An optimal maintenance plan therefore varies if partial replacement is considered to be a part of maintenance efforts or not. This distinction between maintenance and partial replacement has not been made by previous optimal-control models of capital maintenance, and thus deserves emphasis.

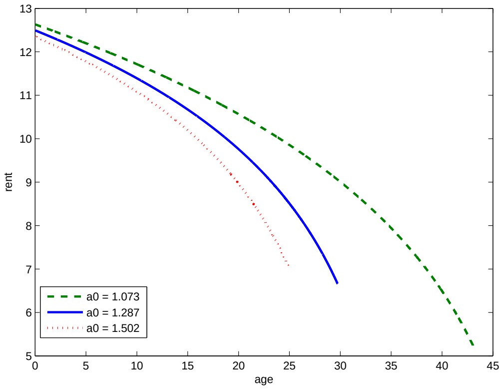
Figure 7 is the plot of the time series of rent, optimal maintenance expenditure and the deterioration rate for various values of the initial condition a_0; that is, when $a_0 = 1.073$ (estimated from the data), when $a_0 = 1.287$ (raised by 20%) and when $a_0 = 1.502$ (raised by 40%). A large a_0 means that an asset is low-tech relative to those of the same vintage. First, we see that the asset is sold approximately at the ages of 43 years, 30 years and 25 years when $a_0 = 1.073$, 1.287 and 1.502 respectively. So, an optimal lifetime is shorter, the larger is a_0. Moreover, just as for an increase in β, an increase in a_0 reduces optimal maintenance expenditure for all $t \in (0, T^*)$, and thus increases the deterioration rate and lowers the rent at every point in time. It is intuitive that the effect of increased a_0 is somewhat similar to that of increased β, since maintenance cannot improve the inherent productivity of the capital asset, just as for the causality between maintenance and obsolescence wherein obsolescence affects the incentives for maintenance but not *vice versa*.

5 Conclusion

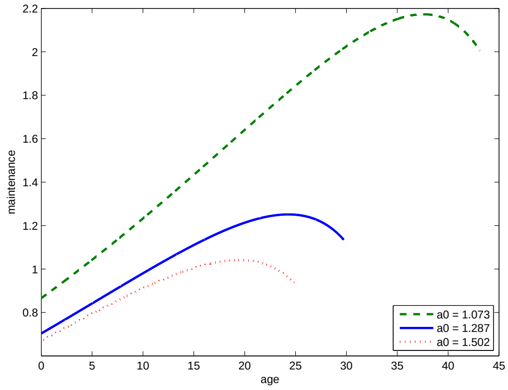
This paper develops an optimal-control model of deterministic capital maintenance, where an indivisible capital asset is used for productive purposes for a length of time and then sold. The production revenue depends on physical wear and tear and obsolescence. Maintenance slows down physical wear and tear but has nothing to do with the effect of obsolescence. An optimal policy is then characterized using the Maximum Principle, phase portraits and a comparative dynamics analysis. Given the characterization, the model is then estimated from U.S. data on rent and maintenance expenditure on office buildings.

We add to the optimal-control literature on deterministic capital maintenance. First, in contrast to previous models in the literature, the model developed here is a nonlinear model in maintenance expenditure, a control variable. Specifically, our model may be viewed as a nonlinear extension of Thompson's model (1968). With this extension, it has been shown that an optimal maintenance policy may vary qualitatively with asset type (*i.e.*, whether an asset wears out rapidly without maintenance or not). The dependence of an optimal maintenance on asset type has not hitherto been fully explored in the literature.

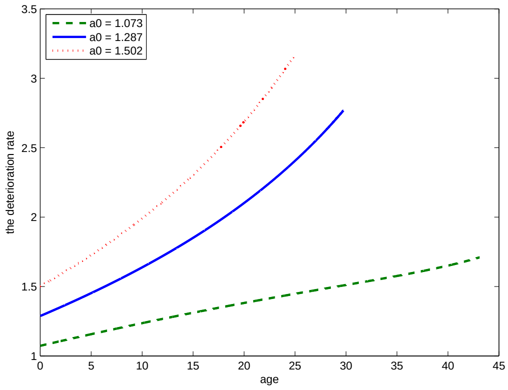
Second, we distinguish between maintenance and partial replacement. In the previous models, maintenance is assumed to be effective at preventing both the process of physical wear and tear and the effects of obsolescence. However, it is argued here that it is partial replacement rather than maintenance that counteracts the effects of obsolescence. By as-



(a) rent



(b) maintenance



(c) the deterioration rate

Figure 7: Estimated time paths for various a_0

suming that maintenance is effective at preventing physical wear and tear but has nothing to do with the effect of obsolescence, we make this distinction explicit. Simulations demonstrate that an increase in physical wear and tear may have a different effect on maintenance expenditure than that of obsolescence. Although both reduce an owner's receipts, an increase in physical wear and tear can induce more maintenance efforts, whereas an increase in obsolescence reduces maintenance incentives all the time. An optimal maintenance policy may vary if partial replacement is considered to be a part of maintenance efforts or not, so the distinction matters.

Appendix: Proofs

Lemma 2. *Suppose that* $\{T^*, m^*, a_T^*\}$ *maximizes* J *in (2) subject to (1). Then, there exist a piecewise smooth function* $\mu(t)$ *and a piecewise continuous Lagrange multiplier func-*

tion $\nu(t)$, such that for all $t \in [0, T^*]$

$$L_m \leq 0, \qquad m^* L_m = 0$$
$$L_{mm} \leq 0$$
$$\nu \geq 0, \qquad \nu L_\nu = 0$$

where L is the Lagrangian function defined by

$$L = L(a, m, \mu, \nu) = H + \nu(a - a_0).$$

Moreover except for the points of discontinuities of m,

$$\dot{a}^* = L_\mu, \qquad \dot{\mu} = r\mu - L_a.$$

In addition, the following transversality conditions are also necessary. At $t = T^*$,

$$\mu \leq S'(a_T^* + b)$$
$$H \geq rS(a_T^* + b) - S'(a_T^* + b)\dot{b}$$

with equality when $a_T^* > a_0$ in the former and when $T^* < Z$ in the latter.

Lemma 2 is the necessary conditions for the problem posed in Section 2. A general statement is in many places. See, for instance, Theorem 6.1 in Caputo (2005) for the necessary conditions for an optimal control problem with inequality state constraints, and Theorems 10.2–10.3 in that book for the salvage-value transversality conditions for a free endpoint and free terminal-date control problem.

Proof of Part (i) of Proposition 1. Applying the former transversality condition in Lemma 2, the latter transversality condition becomes

$$R - m^* + S'(a_T^* + b)\dot{a}^* \geq R - m^* + \mu\dot{a}^* \geq rS(a_T^* + b) - S'(a_T^* + b)\dot{b}.$$

The claim of this proposition is just this rearranged. □

Proof of Part (ii) of Proposition 1. First, note that $z'(m) > 0$ and $L_{mm} \leq 0$ imply $\mu \leq 0$ for all $t \in (0, T)$ in an optimal plan. Suppose $t \in I$. By hypothesis, $a^* > a_0$ at every $t \in I$. So, $\nu L_\nu = 0$ implies $\nu = 0$. Apply this to expand L_a, yielding

$$\dot{\mu} = (r - \alpha)\mu - R'(a^* + b).$$

Let $f(m) \equiv -1 - \mu z'(m) = L_m$, indicating the rate of change in L for a unit increase in m. If $f(0) \leq 0$, then $z''(m) < 0$ implies $f(m) < 0$ for all $m > 0$. That is, $f(0) > f(m)$ for any $m > 0$. Therefore, $m^* = 0$ if $t \in I$ and $1 \geq -\mu z'(0)$.

If $f(0) > 0$ then $\mu < 0$. A small increase in m from zero strictly increases L, implying $m^* > 0$. From $m^* L_m = 0$, this implies $L_m = 0$. Rearranging it gives $-1 = \mu z'(m^*)$.

Next suppose $t \in B$. By definition, $a^* = a_0$ and $\dot{a}^* = 0$ at every $t \in B$. From $\dot{a} = L_\mu$, this implies $\alpha a_0 = z(m^*)$ for all $t \in B$. Note $m^* > 0$, since $z(m) > 0$ if and only if $m > 0$ and since $\alpha a_0 > 0$. From $m^* L_m = 0$, this suggests $L_m = 0$. So, $\mu = -1/z'(m^*)$. Since m^* is fixed, μ is constant and trivially continuous over B. That is, $\dot{\mu} = 0$. Apply these to $\dot{\mu} = r\mu - L_a$ yielding

$$\nu = -(r - \alpha)/z'(m^*) - R'(a_0 + b). \tag{8}$$

Note that $\nu \geq 0$ if and only if $(r-\alpha)+R'(a_0+b)z'(m^*) \leq 0$. Also $\dot{\nu} > 0$ because $R'(a+b) < 0$ and $\dot{b} = \beta > 0$.

Condition (6) is merely the former transversality condition in Lemma 2. □

Proof of Proposition 2. If L is concave in (m, a) and $S(a_T + b)$ is concave in a_T, then from the Mangasarian sufficient conditions the claim follows. See, for instance, Theorem 10.4 in Caputo (2005). A sufficient condition for L to be concave in (m, a) is that the Hessian of L is negative semi-definite. If $D^2 L$ indicates the Hessian of L, then

$$D^2 L = \begin{bmatrix} L_{mm} & L_{ma} \\ L_{am} & L_{aa} \end{bmatrix} = \begin{bmatrix} -\mu z''(m) & 0 \\ 0 & R''(a_T + b) \end{bmatrix}.$$

Since $L_{mm} \leq 0$, $L_{aa} < 0$ and $|D^2 L| \geq 0$, the determinant test establishes that L is concave in (m, a). Also $S_{a_T a_T}(a_T + b) = S''(a_T + b) < 0$. Hence S is concave in a_T. Applying the Mangasarian sufficient conditions completes the proof. □

Proof of Part (i) of Lemma 1. Set $\dot{a}^* = 0$, yielding the $\dot{a}^*{=}0$ isocline $a^* = z(m^*)/\alpha$ in the (m^*, a^*)-space. Since $\alpha > 0$, this is well-defined and increasing (and concave). Since $\lim_{m \to 0^+} z(m) = 0$, the isocline includes the vector $(\lim_{m^* \to 0^+} m^*, \lim_{a^* \to 0^+} a^*)$. Since α is constant, the isocline does not change with time. □

Lemma 3. *Given T^*, suppose $m^* > 0$ and $a^* > a_0$ for all $t \in (0, T^*)$. Then, m^* satisfies $\dot{m}^* = x(m^*) G(m^*, a^*, b)$, where $x(m) = -z'(m)/z''(m)$ and $G(m, a, b) = r - \alpha + R'(a + b) z'(m)$.*

Proof of Lemma 3. By Proposition 1, $1 = -\mu z'(m^*)$ for all $t \in (0, T^*)$. Rearranging it, $\mu = -1/z'(m^*)$ for any $t \in (0, T^*)$. Differentiate the both sides with respect to time, yielding

$$\dot{\mu} = z''(m^*)\dot{m}^*/z'(m^*)^2.$$

Apply these to $\dot{\mu} = r\mu - L_a$ in (5) and rearrange the terms. The claim follows. □

Proof of Part (ii) of Lemma 1. By Lemma 3, $\dot{m}^* = 0$ occurs if and only if $r > \alpha$, since $z' > 0$, $z'' < 0$ and $R' < 0$. So, the $\dot{a}^*$=0 isocline exists if and only if $r > \alpha$. The rest of the claim follows from the lemma below. □

Lemma 4. *Let $r > \alpha$ and let $m^* > 0$ and $a^* > a_0$ for all $t \in (0, T^*)$ given T^*. Suppose that a set $\{(m^*, i(m^*; b))\}$ of vectors in (m^*, a^*)-space indicates the $\dot{m}^*$=0 isocline at time $t \in (0, T^*)$. Then, $i_m(m^*; b) > 0$ and $i_b(m^*; b) < 0$.*

Proof of Lemma 4. Lemma 3 shows $\dot{m}^* = x(m^*)G(m^*, a^*, b)$. Set $G(m^*, a^*, b) = 0$. Note $G_a(m^*, a^*, b) < 0$, since $x(m) > 0$, $z' > 0$ and $R''(a + b) < 0$. Apply the Implicit Function Theorem to rewrite $a^* = i(m^*, b)$ where $G_a(m^*, i(m^*, b), b) = 0$. Then, the set of vectors $\{(m^*, i(m^*, b))\}$ corresponds to the $\dot{m}^* = 0$ isocline at time t. Since $G_m(m^*, a^*, b) > 0$ and $G_b(m^*, a^*, b) < 0$, the Implicit Function Theorem also proves $i_m(m^*; b) = -G_m/G_a > 0$ and $i_b(m^*, b) = -G_b/G_a < 0$. □

Proof of Proposition 3. Draw the phase diagram on $(\dot{m}^*, \dot{a}^*)$ in (m^*, a^*)-space. First, the transition function on a in (1) implies that the area above the $\dot{a}^*$=0-isocline is where a^* increases over time, while the area below is where a^* is decreases over time. Second, Lemma 3 implies $\dot{m}^* \leq 0$ if $r - \alpha < 0$, since $z' > 0$, $z'' < 0$ and $R' < 0$. From these pieces of information, we can draw a phase portrait as in Figure 2(a).

The initial maintenance expenditure is less $z^{-1}(\alpha a_0)$; otherwise, $a^* = a_0$ at the next moment. Since $(m^*(0), a^*(0))$ is above the $\dot{a}^*$=0-isocline, the maintenance expenditure decreases through time and the deterioration rate increases though time at time 0 and also thereafter in an optimal plan.

Differentiate the both sides of the transition function $\dot{a} = \alpha a - z(m)$ with respect to time, yielding

$$\ddot{a} = \alpha\dot{a} - z'(m)\dot{m}.$$

Since a^* increases through time and m^* decreases through time, this suggests $\ddot{a}^* > 0$ for all $t \in (0, T^*)$. □

Proof of Proposition 4. Draw the phase diagram on $(\dot{m}^*, \dot{a}^*)$ in (m^*, a^*)-space. First, the transition function on a in (1) implies that the area above the $\dot{a}^*$=0-isocline is where a^* increases over time, while the area below is where a^* decreases over time. Second, the transition function on m^* in Lemma 3 implies that the area above the $\dot{m}^*$=0-isocline is where m^* decreases over time, while the area below is where m^* increases over time. From these pieces of information, we can draw a phase portrait as in Figure 2(b). Note that the $\dot{a}^*$=0-isocline shifts down as time passes, so its intercept to the a^*-axis at time t can be much larger or smaller.

There are at least three cases to consider. First, suppose that the $\dot{m}^*$=0-isocline has an initial a^*-intercept, ι_0, that is greater than a_0. The initial maintenance outlay is less than $z^{-1}(\alpha a_0)$; otherwise, $a^* = a_0$ at the next moment. So, $\{m^*(0), a^*(0)\}$ is below the $\dot{m}^*$=0-isocline and above the the $\dot{a}^*$=0-isocline, implying that both m^* and a^* are increasing with time early on. If ι_0 is *not* far greater than a_0, then the trajectory of (m^*, a^*), which may be heading to the upper-right, may encounter the $\dot{m}^*$=0-isocline, which is shifting down, at some time $t' \in (0, T^*)$. If so, the value of m^* is the largest at time t' and starts declining beyond time t' while a^* keeps increasing. Alternatively, if ι_0 is far greater than a_0, then the trajectory of (m^*, a^*) may not able to reach the $\dot{m}^*$=0-isocline before time T^*. So, in this case, both m^* and a^* increase monotonically over $(0, T^*)$. There is yet another case. Suppose that the $\dot{m}^*$=0-isocline has an initial m^*-intercept that is greater than $z^{-1}(\alpha a_0)$. In this case, clearly $\{m^*(0), a^*(0)\}$ is above both of the $\dot{m}^*$=0- and the $\dot{a}^*$=0-isoclines. Since the $\dot{m}^*$=0-isocline shifts down as time passes, this implies that m^* decreases with time while a^* increases with time over $(0, T^*)$. The other cases are that the initial intercept of the $\dot{m}^*$=0-isocline to the a^*-axis is between zero and a_0 and that the initial intercept of the $\dot{m}^*$=0-isocline to the m^*-axis is between zero and $z^{-1}(\alpha a_0)$. They may be discussed in a similar fashion to the first case. □

Proof of Proposition 5. We employ the dynamic envelope method proposed by Caputo (2005). To begin, consider the following primal, fixed endpoint and fixed time control problem:

$$V(\omega) \equiv \max_{m \geq 0} \int_0^T e^{-rt} \left[R(a+b) - m\right] \mathrm{d}t \qquad \text{subject to (1)}$$

where $\omega = (\beta, a_0, a_T, T)$. By using the Maximum Principle, the maximized value function V is also written out with the Hamiltonian H as

$$V(\omega) = \int_0^T e^{-rt} \left[H(a^*, m^*, \mu; \omega) - \mu \left[\alpha a^* - z(m^*)\right]\right] \mathrm{d}t.$$

Let $B(\omega')$ be a four-dimensional open ball centered at the given parameter value ω' of radius $\delta > 0$. The dynamic "primal-dual" problem corresponding to this primal problem is defined by

$$\begin{aligned} \max_{\omega} \ \phi(\omega) \equiv & \int_0^T e^{-rt} \left[R(a^*(t;\omega^*) + b(t;\omega)) - m^*(t;\omega^*)\right] \mathrm{d}t - V(\omega) \\ \text{s.t.} \quad & \alpha a^*(t;\omega^*) - z(m^*(t;\omega^*)) - \dot{\mu}(t;\omega^*), \quad a^*(0;\omega^*) = a_0, \quad a^*(T;\omega^*) = a_T \end{aligned}$$

where $(a^*(t;\omega^*), m^*(t;\omega^*))$ is the optimal pair of curves given that the parameter vector ω is fixed at the arbitrary value $\omega^* = (\beta^*, a_0, a_T, T) \in B(\omega')$, and $\mu^*(t;\omega^*)$ is its associated costate function. By construction, $\phi(\omega) \leq 0$ for all $\omega \in B(\omega'')$ with equality when $\omega = \omega^*$.

Noticing that the correct multiplier function for the state equation constraint is $\mu(t;\omega^*)$, form the Lagrangian $\mathcal{L}$ of this dynamic primal-dual problem as

$$\begin{aligned}\mathcal{L}(\omega) &\equiv \int_0^T e^{-rt} \{R(a^*(t;\omega^*) + b(t;\omega)) - m^*(t;\omega^*) + \mu(t;\omega^*)\left[\overline{\alpha} a^*(t;\omega^*) - z(m^*(t;\omega^*))\right. \\ &\quad \left.-\dot{\mu}(t;\omega^*)\right]\} \,\mathrm{d}t - V(\omega) \\ &= \int_0^T e^{-rt} \{H(a^*(t;\omega^*), m^*(t;\omega^*), \mu(t;\omega^*);\omega) - \mu(t;\omega^*)\left[\alpha a^*(t;\omega^*) - z(m^*(t;\omega^*))\right]\} \,\mathrm{d}t \\ &\quad - V(\omega).\end{aligned}$$

Using the integration by parts and then using the initial and terminal condition constraints yield

$$\begin{aligned}\mathcal{L}(\omega) = &\int_0^T e^{-rt} \left[H(a^*(t;\omega^*), m^*(t;\omega^*), \mu(t;\omega^*);\omega) + \dot{\mu}(t;\omega^*)a^*(t;\omega^*)\right] \mathrm{d}t \\ &- \mu(T;\omega^*)a_T + \mu(0;\omega^*)a_0 - V(\omega).\end{aligned}$$

By the first-order necessary condition for the primal-dual problem, $\mathcal{L}_\omega(\omega) \equiv \underline{0}$. So,

$$\mathcal{L}_\beta(\omega) = \int_0^T e^{-rt} H_\beta(a^*(t;\omega^*), m^*(t;\omega^*), \mu(t;\omega^*);\omega)\mathrm{d}t - V_\beta(\omega) \equiv 0. \tag{9}$$

Moreover, the second-order necessary condition together with the fact that β does not appear in the transition function nor in the endpoint constraints on a^* suggests that $\mathcal{L}_{\beta\beta}(\omega)$ be negative semidefinite (n.s.d.) for all $\omega \in B(\omega')$ free of constraint.

Differentiating (9) with respect to β and remembering $b = \beta t$, the negative semidefiniteness of $\mathcal{L}_{\beta\beta}$ implies

$$\mathcal{L}_{\beta\beta} = \int_0^T e^{-rt} R''(a^*(t;\omega^*) + b(t;\omega))t^2 \mathrm{d}t - V_{\beta\beta}(\omega) \leq 0.$$

Since $R'' < 0$ for all $\omega \in B(\omega')$, this in turn implies $V_{\beta\beta}(\omega) \geq 0$ for all $\omega \in B(\omega')$, or that V is convex in β for all $\omega \in B(\omega')$.

From the first-order necessary condition (9) and the convexity of V in β,

$$\begin{aligned}V_{\beta\beta}(\omega) &= \int_0^T e^{-rt} \left[H_{\beta a}(t;\omega^*)\frac{\partial a^*(t;\omega^*)}{\partial\beta} + H_{\beta\beta}(t;\omega^*)\right] \mathrm{d}t \\ &= \int_0^T e^{-rt} R''(a^*(t;\omega^*) + b(t;\omega))\, t^2 \left[\frac{\partial a^*(t;\omega^*)}{\partial\beta} + 1\right] \mathrm{d}t \geq 0,\end{aligned}$$

where $H_{\beta i}(t;\omega^*) = H(a^*(t;\omega^*), m^*(t;\omega^*), \mu(t;\omega^*);\omega^*)$, $i = a$ or β. So, it is ruled out that $\partial a^*(t;\omega^*)/\partial\beta \geq 0$ for all $t \in (0, T^*]$ in an optimal plan. Since the choice of ω^* is arbitrary, this holds for all $\omega \in B(\omega')$.

What remains to show is that if $\partial m^*/\partial\beta \geq 0$ for all $t \in (0, T^*]$ then $\partial a^*/\partial\beta \geq 0$ for all $t \in (0, T^*]$. This is established by contradiction. Suppose $\partial m^*/\partial\beta \geq 0$ for all $t \in (0, T^*]$ but $\partial a^*/\partial\beta < 0$ at some $t' \in (0, T^*]$. To begin, let $t' = T^*$. Then, while the first-order condition (4) insists that increased $m^*(t')$ be associated with more negative $\mu(t')$, the terminal condition (6) suggests that more negative $\mu(t')$ be associated with larger a_T^*. A contradiction. So, t' cannot be T^*. Next, proceeding backwards, let t' be the time a moment before T^*. Then, while condition (4) insists that increased $m^*(t')$ be associated with more negative $\mu(t')$, condition (7), which may be approximated as

$$\mu(t') \approx e^{-(r-\alpha)(T^*-t')} S'(a_T^* + b(T^*)) + R'(a^*(t') + b(t'))(T^* - t'),$$

suggest that more negative $\mu(t')$ be associated with larger $a^*(t')$. A contradiction. So, t' cannot be the time a moment before T^* either. Now, consider any $t' \in (0, T^*)$. Provided $\partial a^*/\partial\beta \geq 0$ for all $t \in (t', T^*]$, it is straightforward to show that conditions (4) and (7) suggest that increased $m^*(t')$ be associated with increased $a^*(t')$. A contradiction. Since t' is arbitrary, there exists no $t' \in (0, T^*]$ such that $\partial a^*/\partial\beta < 0$ while $\partial m^*/\partial\beta \geq 0$ for all $t \in (0, T^*]$. □

Proof of Proposition 6. The sign of the second-order derivative of V with respect to β has been shown in the proof of Proposition 5. With $\omega = (\beta, a_0, T, a_T)$, (9) implies

$$V_\beta(\omega) = \int_0^T e^{-rt} R'(a^*(t;\omega) + b(t;\omega))\, t \,\mathrm{d}t.$$

Since $R' < 0$, this is negative. By redefining ω by $\omega = (\alpha, a_0, T, a_T)$ and repeating the same discussion, we have $\mathcal{L}_\omega(\omega) \equiv \underline{0}$. So,

$$V_\alpha(\omega) \equiv \int_0^T e^{-rt} H_\alpha(a^*(t;\omega), m^*(t;\omega), \mu(t;\omega);\omega)\mathrm{d}t = \int_0^T e^{-rt} \mu(t;\omega) a^*(t;\omega)\mathrm{d}t < 0$$

since $\mu < 0$ and $a^* > a_0 > 0$ for all $t \in (0, T^*)$. □

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