Spatial Competition and Collaboration Networks*

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Abstract

In this paper, we discuss the formation of collaboration networks among firms that are located in a circular city. The model is a two-stage game. In the first stage, firms form collaboration links, and in the second stage the firms engage in price competition. The model in the second stage is a generalization of Salop's (1979) model. We examine pairwise stability of networks and a stochastic network formation process. In addition, we characterize socially efficient networks.

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1 Introduction

An important development in the last three decades is the increase in the number of interfirm collaboration. This is especially evident in international collaboration. Hagedoorn (2002) reports that more than 50% of newly made R&D collaborations during the period 1960-1998 are international ones. This implies that, though firms may incur higher transaction costs in keeping a long-distance collaboration than that in keeping short-distance one, many firms prefer long-distance one.

In this paper, we will theoretically investigate the relationship between firms' strategic incentives to form collaborative relations and location of the firms. To be more precise, we will discuss collaboration networks among firms that are located in a circular city and engage in price competition. Firms can form collaboration links with other firms. If two firms form a link between them, then the production costs of the firms are reduced. A set of all firms and the existing links between them is said to be a network.

Some previous studies discuss Cournot or Bertrand competition with formation of collaboration networks. See, for instance, Goyal and Joshi (2003, 2006), Kawamata (2004), Bloch (2005) and Okumura (2007). Johnson and Gilles (2000) and Jackson and Rogers (2005) discuss the formation of communication networks where each player's location can be different. Both of them extend the connections model introduced by Jackson and Wolinsky (1996). However, there is no previous study analyzing a spatial competition model with formation of collaboration networks among firms. The spatial competition model in this paper is a generalization of Salop's (1979) model.

A network is represented by a set of nodes and edges. The nodes represent the firms and the edges represent the existing bilateral collaboration links between firms. We will analyze the stability and market outcomes of networks. In the previous models of network formation among firms, some market outcomes and stability results of a network is same as those of its renamed network. Consider a six-firm example. Figure 1 depicts two different networks. In a Cournot or Bertrand oligopoly model with network formation, discussed by Goyal and Joshi (2003) for example, stability results of the networks are completely same. Moreover, some market outcomes, e.g., social surplus, of the two networks are same. However, this is not the case in our model. This is because, in the right network, firm 1 forms the link with an adjacent firm (firm 2), but in the left network, firm 1 forms the link with a firm (firm 4) that is not adjacent to firm 1. Thus, stability results of a network may differ from those of the other network. In addition, market outcomes, e.g., social surplus, of a network are different from those of the other network.

Figure 1

In this paper, we will discuss pairwise stability of networks. Pairwise stability is the solution concept introduced by Jackson and Wolinsky (1996). Some of our stability results are closely related to Goyal and Joshi's (2003) results. They discuss a Cournot oligopoly model with network formation and focus on pairwise stability of networks. They show that under general market conditions the complete network is pairwise stable, and with some additional assumption the complete network is uniquely pairwise stable. Some of our stability results are similar to theirs. However, note that our model is not in the class discussed by Goyal and Joshi (2003). They discuss a market in *moderate competition*; that is, lower cost firms make larger profits. On the other hand, our model does not satisfy the moderate competition condition; that is, lower cost firms may make smaller profits. Furthermore, Goyal and Joshi (2003) show that the complete network is the unique socially efficient network. We also show that the complete network is uniquely socially efficient. Okumura (2007) also discusses a Cournot model with network formation and shows that a stochastic network formation process converges to the complete network with positive probability if the number of firms is even. The formation process is introduced by Jackson and Watts (2001) and Watts (2001). We will also show that the formation process may converge to the complete network if the number of firms is even.

In this paper, we attempt to answer why firms prefer long-distance collaborations to short distance ones. We will show that a firm prefers forming a link with the firm located far from each other if the cost-reducing effects are same. Moreover, we will give some specific examples indicating that more stable network has longer average distance between linked firms. Obviously, the distance between two firms in different countries is often much longer than that between two firms in a same country. Thus, this result may explain the fact that more than half of newly established collaborations during the period 1960-1998 are international ones.

In this paper, we will examine a generalized Salop's (1979) circular city model. In the original Salop's model, he assumes that the cost function of each firm is symmetric. Since our model includes the cases that the marginal costs of firms are asymmetric, we will derive the solution of the model that is a generalization of Salop's (1979) model. Economides (1993) firstly introduces the model. However, though his characterization of the solution is sufficient for his analysis, it is insufficient for our analysis. Thus, in this paper, we will completely derive the equilibrium prices of the model.

In Section 2, we will introduce our model. In Section 3, we will derive the equilibrium of a spatial competition model. In Section 4, we will examine stable networks and formation process. Further, the socially efficient network is discussed in Section 4. In Section 5, we will give specific examples of our model and focus on stability, efficiency and the average distance between linked firms. Section 6 concludes.

2 Model

We will discuss a two-stage game. At the first stage, bilateral collaboration links are formed among firms. At the second stage, the firms engage in price competition. Let the finite set of the firms be $N = \{1, 2, \cdots, n\}$. We assume

$n \geq 3$. Let a set of N and the existing collaboration links between two firms in N be a network g. We will write $ij \in g$, indicating that firms i and j are linked in g, while $ij \notin g$ indicates that i and j are not linked in g. In addition, let $g + (-)ij$ be the network obtained from g by adding (severing) a link between i and j to g. The number of links that i forms in g is given by $\eta^i(g)$.

Let $c : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ be a function satisfying $c(m) < c(m-1)$ for all $m \geq 1$. The marginal cost of a firm is assumed to be constant and given by $c^i = c(\eta^i(g))$ for $i \in N$. That is, firm i's marginal cost depends on the number of the firms linked with i.

Each firm i chooses its price p^i to maximize its own profit. The profit of i is given by $(p^i - c^i)D^i$ where D^i is the demand that i faces.

There is a circular city with perimeter 1. In the city, the firms are equidistanly located from each other. Without loss of generality, let the address of firm i be $(i-1)/n$. Let the function $d : N \times N \to \mathbb{Z}_+$ be such that

$$d(i,j) = d(j,i) = \min\{j - i, n - j + i\} \text{ for } j \geq i.$$

That is, $d(i,j)$ indicates the distance between i and j times n. In addition, let

$$\bar{d}_n = \begin{cases} n/2 \text{ if } n \text{ is even,} \\ (n-1)/2 \text{ if } n \text{ is odd.} \end{cases}$$

That is, for any given n, $\bar{d}_n = \max_{i,j \in N} d(i,j)$.

Consumers are uniformly distributed with density 1 around the city. She/He purchases at most one product from a firm. When a consumer living in $s \in [0,1]$ purchases a product from firm i, she/he incurs the transport cost $\|(i-1)/n - s\|\, t$ where $\|(i-1)/n - s\| = \min\{|(i-1)/n - s|, 1 - |(i-1)/n - s|\}$. His/Her utility is given by

$$v - \|(i-1)/n - s\|\, t - p^i \text{ if he/she purchases a product from } i,$$
$$0 \text{ if he/she purchases nothing.}$$

In this paper, we will focus on the case where (a) all consumers purchase a product and (b) each firm sells for some consumers living in both its left and right sides. In equilibrium, (a) is satisfied if v is sufficiently large and (b) is satisfied if t is sufficiently large in any network. Hence we assume that v and t are sufficiently large.

3 Spatial Competition

In this section, we will characterize the equilibrium in the second stage of the game. Thus, we will derive the price equilibrium in a given network g.

A consumer who is indifferent between purchasing from i and $i+1$ lives in $s_i \in [(i-1)/n, i/n]$ satisfying

$$\begin{aligned} v - (s_i - (i-1)/n)\, t - p^i &= v - (i/n - s_i)\, t - p^{i+1}, \\ s_i &= \frac{p^{i+1} - p^i}{2t} + \frac{2i-1}{2n} \end{aligned}$$

Thus,

$$D^i = \frac{p^{i+1} - p^i}{2t} + \frac{p^{i-1} - p^i}{2t} + \frac{1}{n}$$

where $p^0 = p^n$. The first order conditions are, for all $i \in N$,

$$p^i - \frac{p^{i-1}}{4} - \frac{p^{i+1}}{4} = z^i \equiv \frac{1}{2}\left(c^i + \frac{t}{n}\right) > 0. \tag{1}$$

A solution of this problem $p^* = (p^{1*}, p^{2*}, \cdots, p^{n*})$ will be characterized as follows. For $i = 1, \cdots, n$,

$$\begin{aligned} p^{i*} &= \sum_{j \in N} \alpha^n_{d(i,j)} z^j \qquad (2) \\ &= \begin{cases} \alpha^n_0 z^i + \sum_{k=1}^{(n-2)/2} \alpha^n_k (z^{i+k} + z^{i-k}) + \alpha^n_{n/2}\ z^{i-n/2} \text{ if } n \text{ is even,} \\ \alpha^n_0 z^i + \sum_{k=1}^{(n-1)/2} \alpha^n_k (z^{i+k} + z^{i-k}) \text{ if } n \text{ is odd,} \end{cases} \end{aligned}$$

where $z^{n+k} = z^k$ and $z^{-k} = z^{n-k}$ for $k = 0, 1, \cdots, n/2$. Thus, the equilibrium price of each firm is linearly dependent on z^j for all $j = 1, \cdots, n$ and the effect of z^j on the equilibrium price of i is dependent on the distance between i and j. The effect of z^j on i's price is represented by $\alpha^n_{d(i,j)}$.

Note that Economides (1993, pp.246) has already stated this fact. That is, (2) corresponds to the equation (13) of his paper. However, he does not derive $\alpha^n_{d(i,j)}$ for all $i, j \in N$. Since it is necessary to derive $\alpha^n_{d(i,j)}$ for all $i, j \in N$ for our analysis below, we will derive

$$\begin{aligned} \alpha^n &= (\alpha^n_0, \alpha^n_1, \cdots, \alpha^n_{\bar{d}_n}) \\ &= \begin{cases} (\alpha^n_0, \alpha^n_1, \cdots, \alpha^n_{n/2}) \text{ if } n \text{ is even,} \\ (\alpha^n_0, \alpha^n_1, \cdots, \alpha^n_{(n-1)/2}) \text{ if } n \text{ is odd.} \end{cases} \end{aligned}$$

At first, suppose that n is even. Then, by (1) and (2), α^n satisfies

$$\begin{aligned} \alpha^n_0 - \frac{1}{2}\alpha^n_1 &= 1, & (3) \\ -\frac{1}{4}\alpha^n_k + \alpha^n_{k+1} - \frac{1}{4}\alpha^n_{k+2} &= 0 \text{ for } k = 0, \cdots, n/2 - 2, & (4) \\ \alpha^n_{n/2} - \frac{1}{2}\alpha^n_{n/2-1} &= 0. & (5) \end{aligned}$$

The general solution of (4) is, for $l = 2, \cdots, n/2$,

$$\alpha^n_{n/2-l} = \frac{(\alpha^n_{n/2-1} - y\alpha^n_{n/2})x^l}{x - y} - \frac{(\alpha^n_{n/2-1} - x\alpha^n_{n/2})y^l}{x - y} \tag{6}$$

where x, y are the solutions of $-\frac{1}{4}\alpha^2 + \alpha - \frac{1}{4} = 0.$

By (5) and (6),

$$\alpha^n_{n/2-l} = \frac{(2+\sqrt{3})^l + (2-\sqrt{3})^l}{2}\alpha^n_{n/2} \text{ for } l = 1, 2, \cdots, n/2. \tag{7}$$

By (3) and (7), $\alpha^n_{n/2}$ satisfies

$$\left(\frac{(2+\sqrt{3})^{n/2} + (2-\sqrt{3})^{n/2}}{2} - \frac{(2+\sqrt{3})^{n/2-1} + (2-\sqrt{3})^{n/2-1}}{4}\right)\alpha^n_{n/2} = 1 \tag{8}$$

Therefore, α^n is characterized by (7) and (8) if n is even. Note that, since α^n satisfies (3), (4) and (5), we have

$$\alpha^n_0 + 2\sum\nolimits_{k=1}^{(n-2)/2} \alpha^n_k + \alpha^n_{n/2} = 2. \tag{9}$$

Similarly, if n is odd, then

$$\begin{aligned} \alpha^n_0 - \frac{1}{2}\alpha^n_1 &= 1, & (10)\\ -\frac{1}{4}\alpha^n_k + \alpha^n_{k+1} - \frac{1}{4}\alpha^n_{k+2} &= 0 \text{ for } k = 0, \cdots, (n-1)/2-2, & (11)\\ \frac{3}{4}\alpha^n_{(n-1)/2-1} - \frac{1}{4}\alpha^n_{(n-1)/2} &= 0. & (12) \end{aligned}$$

By (10), (11) and (12), for $l = 1, 2, \cdots, (n-1)/2$

$$\alpha^n_{(n-1)/2-l} = A(l)\alpha^n_{(n-1)/2}, \tag{13}$$

$$\left(A((n+1)/2) - \frac{A((n+1)/2-1)}{2}\right)\alpha^n_{(n-1)/2} = 1, \tag{14}$$

$$\text{where } A(l) = \frac{(\sqrt{3}+1)(2+\sqrt{3})^l + (\sqrt{3}-1)(2-\sqrt{3})^l}{2\sqrt{3}},$$

are satisfied. Note also that

$$\alpha^n_0 + 2\sum\nolimits_{k=1}^{(n-1)/2} \alpha^n_k = 2. \tag{15}$$

Thus, we have the following solution;

Proposition 1 *The equilibrium price vector p^* is, for each $n \geq 3$ and $i = 1, \cdots, n$,*

$$p^{i*} = \frac{t}{n} + \frac{1}{2}\sum_{j \in N} \alpha^n_{d(i,j)} c^j \tag{16}$$

where α^n is characterized by (7), (8), (13) and (14).

A parameter $\alpha^n_{d(i,j)}$ represents the effect on the equilibrium price of firm i of the marginal cost of firm j. We immediately have the following result on α^n.

Remark 1 *Parameters of the equilibrium price (16) satisfy* $2 > \alpha_0^n > 1 > \alpha_1^n > \cdots > \alpha_{d_n}^n > 0$ *for all* n.

(Proof) By (7), (8), (13) and (14), we have $\alpha_0^n > \alpha_1^n > \cdots > \alpha_{d_n}^n > 0$. By (3) and (10), and $\alpha_1^n > 0$, we have $\alpha_0^n > 1$. Next, by (9), (15) and $\alpha_l^n > 0$ for all l, we have $2 > \alpha_0^n$. Finally, by (9) and (15), $\alpha_0^n > 1$ and $\alpha_l^n > 0$ for all l, we have $1 > \alpha_1^n$. (Q.E.D.)

This indicates that the marginal production costs of all firms affect the price of each firm. In addition, the effect on the price of $i \in N$ of the marginal cost of $j \in N$ is larger than that of the marginal cost of $k \in N$ if and only if i is located nearer to j than to k.

4 Networks

4.1 Stability and Formation Process

We will analyze the first stage of our model. That is, we will discuss the formation of collaboration networks. Since the price of each firm is dependent on the marginal costs of all firms, the prices depend on the network g. Thus, we will write the equilibrium price vector as $p^*(g) = (p^{1*}(g), p^{2*}(g), \cdots, p^{n*}(g))$. The profit of a firm is also dependent on g. Let the profit function of firm i be

$$\Pi^i(g) = D(p^{i-1*}(g), p^{i*}(g), p^{i+1*}(g))(p^{i*}(g) - c(\eta^i(g)).$$

Since (1) holds, the demand of firm i in g is

$$D(p^{i-1*}(g), p^{i*}(g), p^{i+1*}(g)) = \frac{1}{t}(p^{i*}(g) - c(\eta^i(g)).$$

Therefore,

$$\Pi^i(g) = \frac{1}{t}(p^{i*}(g) - c(\eta^i(g)))^2,$$

and

$$\begin{aligned} &\text{sign}[\Pi^i(g+ij) - \Pi^i(g)] = \\ &\text{sign}[p^{i*}(g+ij) - c(\eta^i(g)+1) - \{p^{i*}(g) - c(\eta^i(g))\}]. \end{aligned} \tag{17}$$

By Proposition 1, we have

$$\begin{aligned} &p^{i*}(g+ij) - c(\eta^i(g)+1) - [p^{i*}(g) - c(\eta^i(g))] = \\ &\left(1 - \frac{1}{2}\alpha_0^n\right)\left(c(\eta^i(g)) - c(\eta^i(g)+1)\right) - \frac{1}{2}\alpha_{d(i,j)}^n\left(c(\eta^j(g)) - c(\eta^j(g)+1)\right) \\ &\equiv f(n, \eta^i(g), \eta^j(g), d(i,j)). \end{aligned} \tag{18}$$

By (17), we have

$$\text{sign}[f(n, \eta^i(g), \eta^j(g), d(i,j))] = \text{sign}[\Pi^i(g+ij) - \Pi^i(g)].$$

That is, i wants to add the link with j if and only if $f(n, \eta^i(g), \eta^j(g), d(i,j)) > 0$. In addition, i wants to sever the link with j if and only if $f(n, \eta^i(g-ij), \eta^j(g-ij), d(i,j)) < 0$. Thus, firm i's incentive to add/sever the link between i and j is identified by the number of firms in the market, the numbers of the links that i and j have already formed and the distance between the two firms.

We have the following two results.

Proposition 2 *If $c(m')-c(m'+1) = c(m'')-c(m''+1)$ for $m', m'' = 0, 1, \cdots, \bar{d}_n$, then $f(n, m, m', \theta) > f(n, m, m'', \theta - 1)$ for all $m = 0, 1, \cdots, n-2$ and $\theta = 2, 3, \cdots, \bar{d}_n$.*

This result is obvious from (18) and Remark 1. Consider three firms i, l and m. Suppose that $d(i,l) < d(i,m)$; that is, the distance between i and l is shorter than the distance between i and m, and cost-reducing effects of il and im are same. Suppose that $il, im \notin g$. Proposition 2 implies that, if firm i wants to add il, then i also wants to add im. In addition, suppose that $il, im \in g$. If i wants to sever im, then i also wants to sever il. This is because the effect on the equilibrium price of i of cost-reduction of firm m is smaller than that of cost-reduction of firm l. Thus, Proposition 2 means that a firm prefers a long-distance collaboration to short-distance one if the cost-reducing effects are same.

Lemma 1 *Consider two firms $i, j \in N$. If $c(\eta^i(g)) - c(\eta^i(g)+1) = c(\eta^j(g)) - c(\eta^j(g)+1)$, then $f(n, \eta^i(g), \eta^j(g), d(i,j)) > 0$ and $f(n, \eta^j(g), \eta^i(g), d(i,j)) > 0$.*

(Proof) Suppose that $c(\eta^i(g)) - c(\eta^i(g)+1) = c(\eta^j(g)) - c(\eta^j(g)+1) = \Delta$. Then, by (18),

$$\begin{aligned} f(n, \eta^i(g), \eta^j(g), d(i,j)) &> 0 \Leftrightarrow \\ \left(1 - \frac{1}{2}\alpha_0^n - \frac{1}{2}\alpha_{d(i,j)}^n\right)\Delta &> 0. \end{aligned}$$

By (3), (10) and Remark 1,

$$1 - \frac{1}{2}\alpha_0^n - \frac{1}{2}\alpha_{d(i,j)}^n > 0 \text{ for all } d(i,j) = 1, 2, \cdots, \bar{d}_n.$$

Thus, we have Lemma 1. (Q.E.D.)

This result implies that, in the case that the effects of the link between i and j on the marginal costs of i and j are same, i and j want to add ij if $ij \notin g$ and neither i nor j wants to sever ij if $ij \in g$.

We will discuss stability of networks and a stochastic network formation process. We will consider the following two concepts.

The network g is said to be *pairwise stable*, if the following two conditions are satisfied: (a) for all firms i, j $(i > j)$ such that $ij \in g$, $f(n, \eta^i(g-ij), \eta^j(g-ij), d(i,j)) \geq 0$ and $f(n, \eta^j(g-ij), \eta^i(g-ij), d(i,j)) \geq 0$ and, (b) for all firms i, j $(i > j)$ such that $ij \notin g$, if $f(n, \eta^i(g), \eta^j(g), d(i,j)) > 0$, then $f(n, \eta^j(g), \eta^i(g), d(i,j)) < 0$.

Next, we will present the stochastic network formation process introduced by Jackson and Watts (2001) and Watts (2001). We consider a discrete set of points in time $\{1, 2, \cdots, t, \cdots\}$. The network formation starts from the empty network, that is, g_1 is the network with no link. At each time a pair of firms is randomly identified with positive probability. The network at the end of period t is given by g_t. Suppose that firms i and j are identified at $t+1$ and $ij \in g_t$. If either $f(n, \eta^i(g_t - ij), \eta^j(g_t - ij), d(i,j)) < 0$ or $f(n, \eta^j(g_t - ij), \eta^i(g_t - ij), d(i,j)) < 0$, then $g_{t+1} = g_t - ij$. If otherwise, $g_{t+1} = g_t$. Suppose that firms l and m are identified at $t+1$ and $lm \notin g_t$. If both $f(n, \eta^l(g_t), \eta^m(g_t), d(l,m)) > 0$ and $f(n, \eta^m(g_t), \eta^l(g_t), d(l,m)) > 0$, then $g_{t+1} = g_t + lm$. If otherwise, $g_{t+1} = g_t$. If in $g_T = \bar{g}$ no pair of firms will add or sever the link, that is, $\bar{g} = g_T = g_{T+1} = g_{T+2} = \cdots$, then this process converges to $\bar{g}$.

It is obvious that the stochastic formation process converges to only a pairwise stable network.[1] However, there may exist some pairwise stable network to which the formation process converges with probability zero. Jackson and Watts (2001), Kawamata (2004) and Okumura (2007) give some examples.

By Proposition 2 and Lemma 1, we will provide a characterization of pairwise stable networks. Let $N_\eta(g)$ be a set of the firms that have η links in g. In addition, we denote that

$$\begin{aligned} \bar{d}_{lm} &= \max\{d(i,j) \,|\, i \in N_l(g),\ j \in N_m(g) \text{ and } ij \notin g\}, \\ \underline{d}_{lm} &= \min\{d(i,j) \,|\, i \in N_l(g),\ j \in N_m(g) \text{ and } ij \in g\}. \end{aligned}$$

Then, a pairwise stable network is characterized as follows.

Theorem 1 *A network g is pairwise stable if and only if (1) for each $\eta \leq n-1$ and $i, j \in N_\eta(g)$ such that $i \neq j$, $ij \in g$, (2) for each $l, m \leq n-1$ such that $l \neq m$, if there exist some $i \in N_l(g)$ and $j \in N_m(g)$ such that $ij \notin g$ and $f(n, l, m, \bar{d}_{lm}) > 0$, then $f(n, m, l, \bar{d}_{lm}) < 0$, (3) for each $l, m \leq n-1$ such that $l \neq m$, if there exist some $i \in N_l(g)$ and $j \in N_m(g)$ such that $ij \in g$, then $f(n, l-1, m-1, \underline{d}_{lm}) \geq 0$ and $f(n, m-1, l-1, \underline{d}_{lm}) \geq 0$.*

(Proof) First, we will consider the necessity part. By Lemma 1, if $i, j \in N_\eta(g)$ and $ij \notin g$, then both i and j agree to add ij because $c(\eta^i(g)) - c(\eta^i(g)+1) = c(\eta^j(g)) - c(\eta^j(g)+1)$. Thus, the first condition is necessary. The second and third conditions are obviously necessary.

Second we will consider the sufficiency part. By Lemma 1, for each $i, j \in N_\eta(g)$, $ij \in g$ is not severed. By Proposition 2, if either $f(n, l, m, \bar{d}_{lm}) \leq 0$ or $f(n, m, l, \bar{d}_{lm}) < 0$ is satisfied, then for any two firms $i \in N_l(g)$ and $j \in N_m(g)$ such that $ij \notin g$, either i or j disagrees to form ij. Moreover, by Proposition 2, if $f(n, l-1, m-1, \underline{d}_{lm}) \geq 0$ and $f(n, m-1, l-1, \underline{d}_{lm}) \geq 0$, then for any two firms $i \in N_l(g)$ and $j \in N_m(g)$ such that $ij \in g$, neither i nor j wants to sever ij.(Q.E.D)

[1] The fomation process may converge to no network. In that case, the formation process will be in a closed cycle. See Jackson and Watts (2002) with regard to this point.

By Theorem 1, we have the following result.

Corollary 1 *The complete network is pairwise stable.*

This result is direct from Theorem 1. Since $\eta^i(g) = n - 1$ for all $i \in N$ if g is the complete network, then no link will be severed.

Goyal and Joshi (2003, Theorem 3.1) have a similar result, but our model is not in the class of their model.

Remark 2 *In this model, Assumption MC (Goyal and Joshi (2003, p.69)) is not satisfied. That is,* $\Pi^i(g) \neq \Pi^j(g)$ *may be satisfied even if* $c^i(\eta^i(g)) = c^j(\eta^j(g))$*, and* $\Pi^i(g) \leq \Pi^j(g)$ *may also be satisfied even if* $c^i(\eta^i(g)) < c^j(\eta^j(g))$*.*

Consider the model where the firms are equidistantly located on the circle; that is, $a^i = (i-1)/n$ for all $i \in N$. Suppose that $n = 6$, $t = 50$ and $c(0) = 10$, $c(1) = 9$, $c(2) = 1$. If $(\eta^1(g), \eta^2(g), \eta^3(g), \eta^4(g), \eta^5(g), \eta^6(g)) = (1, 2, 1, 0, 2, 2)$, then

$$(\Pi^1(g), \Pi^2(g), \Pi^3(g), \Pi^4(g), \Pi^5(g), \Pi^6(g)) \approx (0.61, 2.52, 0.9, 0.71, 2.18, 2.06).$$

Hence our model does not satisfy Assumption MC.

In addition, the following result is straightforward from Lemma 1.

Corollary 2 *If* $c(\eta^i(g)) = \gamma_0 - \gamma\eta^i(g)$ *where* $\gamma_0 > (n-1)\gamma > 0$*, then the complete network is uniquely pairwise stable.*

In this case, $c(m) - c(m+1) = \gamma$ for all $0 \leq m < n - 1$. Thus, by Lemma 1, we have the result. Goyal and Joshi (2003, Proposition 3.1) also show that, in a Cournot oligopoly model, the complete network is uniquely stable if marginal costs are linearly declining in the number of links.

Finally, we will focus on the stochastic dynamic process defined above. The following result is direct from Lemma 1 and a result of Okumura (2007, Theorem 1).

Corollary 3 *If* n *is even, then the dynamic network formation process converges to the complete network with positive probability.*

The proof is due to that of Okumura (2007, Theorem 1). On the other hand, the formation process may converge to the complete network with probability zero if n is odd. Suppose $c(m) - c(m+1) = \gamma > 0$ for all $1 \leq m < n - 1$ and $c(0) - c(1) > \alpha^n_{d_n}\gamma/(2 - \alpha^n_0)$. This cost function means that the cost reduction effect of adding the first link is very high but subsequent links are not so important. In this case, the formation process converges to the complete network with probability zero if n is odd. See Okumura (2007, p.138) with regard to this point in detail.

4.2 Socially Efficient Network

Next, we will analyze the socially efficient network, in which the social surplus is higher than that in any other networks. We have the following result.

Proposition 3 *If the firms are located geographically equidistant from one another, then the complete network is the uniquely socially efficient network.*

(Proof) In the second stage of this model, we focus on the case where each consumer purchases a product in any network. Thus, in the socially efficient network, the sum of the total production cost and the total transport cost is lower than that in any other networks.

In the complete network, the marginal production cost of all firms is $c(n-1)$. Thus, the total production cost in the complete network is smaller than that in any other network. Further, by Proposition 1, in the complete network, the prices of all firms are equal to $t/n + c(n-1)$. Thus, in the complete network, all consumers purchase from the nearest firm. Therefore, the total transportation cost in the complete network is smaller than or equal to that in any other network. Hence the complete network is uniquely socially efficient. (Q.E.D)

By Proposition 3 and Corollaries 1 and 3, the uniquely socially efficient network is pairwise stable and the formation process converges to the efficient network with positive probability if n is even. Moreover, if marginal costs are linearly declining in the number of links, the complete network is the uniquely socially efficient network and the uniquely pairwise stable network. This fact is straightforward from Proposition 3 and Corollary 2.

5 Six-firm Example

In this section, we will discuss our model in the case of $n = 6$. Stability conditions and social surpluses of some specific networks will be examined. In addition, we will focus on the average distance between linked firms. The average distance in g is defined as $L(g)$. Since the firms are assumed to be located geographically equidistant from one another, the average geographic distance is in direct proportion to the average distance of our definition.

Note that the sentence that g is more (less) stable than g' implies that the stability condition on g is weaker (harder) than that on g'. Likewise, the sentence that g is more (less) efficient than g' means that the social surplus in g is larger (smaller) than that in g'.

First, we focus on a network with two complete groups: one is of four firms and the other is of two firms. See, the networks in Figure 2, for example. The networks can be categorized into three kinds. That is, the stability condition, the social surplus and the average distance between linked firms of a network are equal to those of either g^A, g^B or g^C in Figure 2.

Figure 2

By using Theorem 1, let us derive the stability condition of each network in Figure 2. At first, consider g^A. The network g^A is pairwise stable if and only if either (a) firm 1 (or 4) has no incentive to add the link with 2 (or 3 or 4 or 5); namely $f(6,1,3,2) < 0$, or (b) firm 2 (or 3 or 4 or 5) has no incentive to add the link with 1 (or 4); namely $f(6,3,1,2) < 0$. Since

$$\alpha_0^6 = \frac{52}{45},\ \alpha_1^6 = \frac{14}{45},\ \alpha_2^6 = \frac{4}{45},\ \alpha_3^6 = \frac{2}{45},$$

by (7) and (8), the condition is

$$\begin{aligned} f(6,1,3,2) &= \frac{19}{45}(c(1)-c(2)) - \frac{2}{45}(c(3)-c(4)) < 0 \text{ or} \\ f(6,3,1,2) &= \frac{19}{45}(c(3)-c(4)) - \frac{2}{45}(c(1)-c(2)) < 0, \\ &\Leftrightarrow \frac{19}{2} < \frac{c(3)-c(4)}{c(1)-c(2)} \text{ or } \frac{2}{19} > \frac{c(3)-c(4)}{c(1)-c(2)}. \end{aligned} \tag{19}$$

On the other hand, the stability condition of g^B is that the link between 1 and 4 (or 3 and 6) will not be added. That is, the condition is

$$\begin{aligned} f(6,1,3,3) &= \frac{19}{45}(c(1)-c(2)) - \frac{1}{45}(c(3)-c(4)) < 0 \text{ or} \\ f(6,3,1,3) &= \frac{19}{45}(c(3)-c(4)) - \frac{1}{45}(c(1)-c(2)) < 0, \\ &\Leftrightarrow 19 < \frac{c(3)-c(4)}{c(1)-c(2)} \text{ or } \frac{1}{19} > \frac{c(3)-c(4)}{c(1)-c(2)}. \end{aligned} \tag{20}$$

Similarly, the stability condition of g^C is that (20) is satisfied. As a result, g^A is more stable than g^B and g^C.

Next, we will derive the social surplus in each network: g^A, g^B and g^C. Let the social surplus of a network g be $W(g)$. Then, we have

$$W(g^A) > W(g^B) > W(g^C).$$

The social surplus in g^A is the largest of the three networks. This is because the quantity of the firms that have $c(1)$ in g^A is smaller than those in other networks. That is, since the adjacent firms of a firm with $c(1)$ have $c(3)$ in g^A, many consumers will purchase from the firms with $c(3)$.[2] Similarly, we have the intuition on $W(g^B) > W(g^C)$.

Finally, the average distance of the linked firms in each network is the following order:

$$L(g^A) > L(g^B) > L(g^C).$$

Thus, g^A is the most stable and the most efficient network among the networks such that one complete group consists of two firms and the other complete

[2] Of course, there is a loss of the transportation costs of the consumers in each network. The loss of the transportation costs are largest in g^A. But, we can ignore the loss, because the loss is quite small compared with the loss from high production costs.

group consists of four firms. On the other hand, the stability conditions of g^B and g^C are same, but social surpluses are different; that is, g^B is more efficient than g^C. Thus, there may exist some conflict between stability and efficiency of networks; that is, more efficient network may not be more stable. In this example, more stable network (more efficient network) has longer average distance between linked firms.

Next, we will discuss stability of some more complicated networks. For example, the networks in Figure 3.

Figure 3

By using Theorem 1, we will derive the necessary and sufficient condition that a network in Figure 3 is pairwise stable. First, g^D is pairwise stable if and only if (D1) $f(6,3,4,1) \geq 0$, (D2) $f(6,4,3,1) \geq 0$, (D3) $f(6,0,4,3) \geq 0$, (D4) $f(6,4,0,3) \geq 0$ and (D5) $f(6,1,4,2) < 0$ or $f(6,4,1,2) < 0$. Second, g^E is pairwise stable if and only if (E1) $f(6,3,4,1) \geq 0$, (E2) $f(6,4,3,1) \geq 0$, (E3) $f(6,0,4,2) \geq 0$, (E4) $f(6,4,0,2) \geq 0$ and (E5) $f(6,1,4,3) < 0$ or $f(6,4,1,3) < 0$. Finally, g^F is pairwise stable if and only if (F1) $f(6,3,4,1) \geq 0$, (F2) $f(6,4,3,1) \geq 0$, (F3) $f(6,0,4,2) \geq 0$, (F4) $f(6,4,0,2) \geq 0$ and (F5) $f(6,1,4,3) < 0$ or $f(6,4,1,3) < 0$. By comparing these conditions, g^D is more stable than g^E and g^F, and g^E is more stable than g^F.[3] In addition, we obviously have

$$L(g^D) > L(g^E) > L(g^F).$$

Thus, in these networks, more stable network has longer average distance between linked firms.

6 Extension

In the previous sections, the transport cost of consumers assumed to be linear. In this section, we will consider quadratic transport costs and show that some results in the previous sections continue to hold. That is, the transport cost is given by $(\|(i-1)/n - s\|)^2 t$.

Then, the first order conditions will be, for all $i \in N$,

$$p^i - \frac{p^{i-1}}{4} - \frac{p^{i+1}}{4} = z^{i\prime} \equiv \frac{1}{2}\left(c^i + t\right).$$

Proposition $1'$ *The equilibrium price vector p^* is, for each $n \geq 3$ and $i = 1, \cdots, n$,*

$$p^{i*} = t + \frac{1}{2}\sum_{j \in N} \alpha^n_{d(i,j)} c^j$$

where α^n is characterized by (7), (8), (13) and (14).

[3] The conditions (D1) to (F1) are same. (D2) to (F1) are also same conditions. By Lemma 1, (D3) is weaker than (E3) and (F3), and (E3) is weaker than (F3). Also, (D4) is weaker than (E4) and (F4), and (E4) is weaker than (F4). Finally, (E5) and (F5) are same, and by Lemma 1, (D5) is weaker than (E5) and (F5).

Thus, Propositions 2 and 3, Lemma 1, Theorem 1, and Corollaries 1 to 3 are continue to hold if the transport cost incurred by consumers are quadratic.

7 Concluding Remarks

We discuss collaboration networks among firms located on Salop's (1979) circular city. In our model, each firm noncooperatively decides its price, but firms may form cooperation links in order to reduce their production costs. In previous studies such as Goyal and Joshi (2003) and Okumura (2007), they also discuss similar models. Though our model is not in the class of their model, some results are similar to theirs (see Remark 2 and Corollaries 1 to 3). Moreover, we show that the complete network is socially efficient network.

Moreover, we attempt to answer a question: why firms prefer long-distance collaborations to short-distance ones. In our model, we show that firms prefer long-distance collaborations to short-distance ones if the cost-reducing effects are same (see Proposition 2). In addition, we focus on stable networks and analyze some examples in Section 5. In the examples, we show that more stable network has longer average distance between linked firms. This implies that long-distance collaboration links are likely to form.

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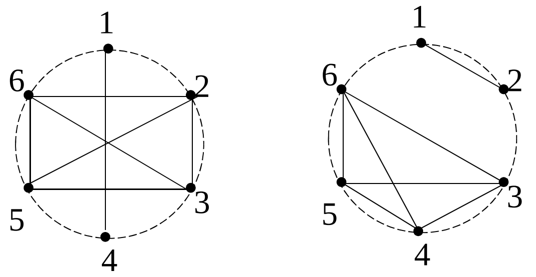

Figure 1: Two Different Networks

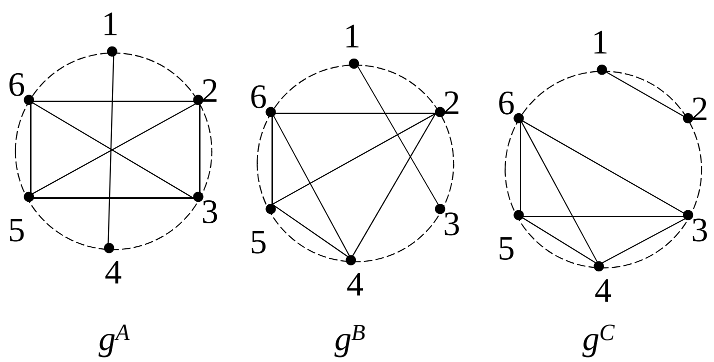

Figure 2: Stable Networks

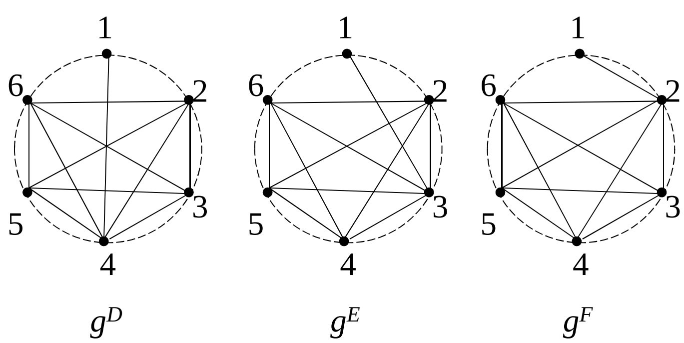

Figure 3: Stable Networks