Noncooperative Coalitional Bargaining with Generalized Selection of Proposers*

Tomohiko Kawamori†

Graduate School of Economics, University of Tokyo

May 8, 2006

Abstract

This paper presents a noncooperative coalitional bargaining model with generalized selection of proposers, which includes Chatterjee *et al.*'s (1993) fixed-order-proposer model and Okada's (1996) random-proposer model as two extreme special cases. In the model, at each round, proposers are selected according to recognition probabilities, which depend on who rejects a proposal at the previous round. This paper provides conditions for the subgame efficiency, which means that the grand coalition is formed and no delay occurs in every subgame, both with each discount factor $\delta < 1$ and in the limit $\delta \to 1$. It is shown that for any discount factor $\delta < 1$, the subgame efficiency is more difficultly achieved as each player is selected as a proposer with a higher probability after she rejects a proposal at the previous round, and in the limit $\delta \to 1$, the probability does not matter for the subgame efficiency.

Keywords: Selection of proposers; Recognition probability; Protocol; Efficiency; Coalitional bargaining

JEL Classification: C78; C72; C73

*The author is grateful to Hiroaki Ino, Masataka Iwata, Minoru Kitahara, Toshihiro Matsumura, Masahiro Okuno-Fujiwara, Yohei Sekiguchi, Kazuo Yamaguchi and Dai Zusai for their valuable comments. Also, the author gratefully acknowledges a Japan Society for the Promotion of Science (JSPS) Research Fellowship for Young Scientists and a Grant-in-Aid for JSPS Fellows of the Ministry of Education, Culture, Sports, Science and Technology, Government of Japan.

†JSPS Research Fellow. `mailto:ee47008@mail.ecc.u-tokyo.ac.jp`.

1 Introduction

Many papers have analyzed noncooperative coalitional bargaining games. Among them, Chatterjee *et al.* (1993) and Okada (1996) are significant. These papers formulate noncooperative dynamic coalitional bargaining with n players, transferable utilities and complete information: In both models, the game is defined based on a characteristic function form game (N, v). At each round, a player becomes a proposer and proposes a coalition and a payoff distribution for the coalition. Then, each player in the proposed coalition announces accepting or rejecting the proposal sequentially until a player rejects it or all players accept it. If all players in the coalition accept the proposal, then the coalition becomes binding with the distribution and the game goes to a new round with the other players. Otherwise, the game goes to the next round with all players remaining. The game proceeds in such a manner. Each player's payoff is the pie distributed to her. Each player discounts future payoffs by the common discount factor δ.

The difference between their models is the procedure in selecting proposers. Chatterjee *et al.*'s (1993) model is called the *fixed-order-proposer (FOP) model*: An order is predetermined on the set of players and players announces accepting or rejecting the proposal sequentially according to the order. At each round, the rejector at the previous round becomes a proposer. On the other hand, Okada's (1996) model is called the *random-proposer (RP) model*: Proposers are randomly selected at every round.

This paper presents a noncooperative bargaining game with generalized selection of proposers. We call the model the *generalized-selection-proposer (GSP) model*: In the model, each round is classified by state (S, s), where S denotes the active player set at the round and s denotes the rejector at the previous round. At each round with state (S, s), a proposer is recognized according to predetermined probability profile $\left(P^S_{s,k}\right)_{k \in S}$, i.e., player i becomes a proposer with probability $P^S_{s,i}$.

The GSP model includes the FOP model and the RP model as two special cases:[1] If

[1] This paper's model also includes Selten's (1981) model and Okada's (1993) model. In terms of the selection of proposers, Selten's (1981) model and Okada's (1993) model are the same as Chatterjee *et al.*'s (1993) model and Okada's (1996) model, respectively. However, in Selten's (1981) model and Okada's (1993) model, the game ends once one coalition is formed. In this paper's model, after a coalition is formed, the game continues with probability π and ends with probability $1 - \pi$.

$P^S_{s,s} = 1$ for any (S, s), the GSP model is equivalent to the FOP model. If $P^S_{s,i} = \frac{1}{|S|}$ for any (S, s) and any $i \in S$, it is equivalent to the RP model. Moreover, as shown in the following section, the FOP model and the RP model are two extreme special cases.

Chatterjee *et al.* (1993) and Okada (1996) investigate the efficiency, i.e., the grand-coalition formation and no delay, in the limit $\delta \to 1$. This paper also considers the efficiency, more precisely the subgame efficiency, which is defined according to Okada. The subgame efficiency means that the full coalition is formed and no delay occurs in any subgame.

As in Chatterjee *et al.* (1993) and Okada (1996), this paper considers the efficiency in the limit $\delta \to 1$. It is shown that in the limit $\delta \to 1$, there exists a subgame efficient equilibrium if and only if $\frac{v(S)}{\sum_{k\in S} P^S_{k,k}} \geq \frac{v(S')}{\sum_{k\in S'} P^S_{k,k}}$ for all positive-worth subsets S and S' of N such that $S \supset S'$ (Theorem 3). If $P^S_{s,s} = P^S_{s',s'}$ for all $s, s' \in S$, the condition is equivalent to Chatterjee *et al.*'s (1993) condition and Okada's (1996), respectively. Theorem 3 implies that in the limit $\delta \to 1$, as long as $\frac{P^S_{s,s}}{\sum_{k\in S} P^S_{k,k}}$ is constant, the difficulty of the subgame efficiency does not depend on the greatness of $P^S_{s,s}$ (Theorem 4). Thus, as Okada points out, the FOP model and the RP model have the same condition for the subgame efficiency in the limit $\delta \to 1$.

On the other hand, this paper investigates the subgame efficiency also for each $\delta < 1$. It is shown that for any discount factor δ, there exists a subgame efficient equilibrium if and only if $\frac{v(S)}{\delta \sum_{k\in S} P^S_{k,k} + (1-\delta)} \geq \frac{v(S')}{\delta \sum_{k\in S'} P^S_{k,k} + (1-\delta)}$ for all positive-worth subsets S and S' of N such that $S \supset S'$ (Theorem 1). In contrast with the limit $\delta \to 1$, Theorem 3 implies that for any discount factor $\delta < 1$, as long as $\frac{P^S_{s,s}}{\sum_{k\in S} P^S_{k,k}}$ is constant, the greater $P^S_{s,s}$ is, the more difficultly the subgame efficiency is achieved (Theorem 2). From this, the subgame efficiency is more easily achieved in the RP model than the FOP model for any $\delta < 1$.

The paper is organized as follows: Section 2 defines a noncooperative coalitional bargaining model, Section 3 investigates the efficiency, and Section 4 concludes the paper.

2 Model

In this section, we describe the GSP model.

Take a characteristic function form game (N, v).[2] Suppose that $v(S) \geq 0$ for all $S \in 2^N$.

[2] N is an arbitrary nonempty finite set and v is an arbitrary mapping from 2^N to $\mathbb{R}$ such that $v(\emptyset) = 0$.

Suppose that (N, v) is superadditive, i.e., for all $S, T \in 2^N$, if $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$.[3] Suppose that $N \not\ni 0$.

We define noncooperative bargaining games based on the above cooperative game. Let $\mathfrak{S} \equiv \{S \in 2^N \mid v(S) > 0\}$. For $S \in \mathfrak{S}$, let $S^* \equiv \{0\} \cup S$. For $S \in \mathfrak{S}$, let[4]

$$\Delta^S \equiv \left\{ P \in \mathbb{R}_+{}^{S^* \times S} \mid \forall s \in S^*, \sum_{k \in S} P(s,k) = 1 \wedge \forall s \in S, P(s,s) > 0 \right\}.$$

For any $S \in \mathfrak{S}$, any $P \in \Delta^S$ and any $(s,k) \in S^* \times S$, let $P_{s,k} \equiv P(s,k)$. For any $P \in \bigcup_{\hat{S} \in \mathfrak{S}} \Delta^{\hat{S}}$, for any $S' \subset S$, let $\operatorname{tr}_{S'} P \equiv \sum_{k \in S'} P_{k,k}$, where S is the unique set that satisfies $P \in \Delta^S$. For any $P \in \bigcup_{\hat{S} \in \mathfrak{S}} \Delta^{\hat{S}}$, let $\operatorname{tr} P \equiv \operatorname{tr}_S P$, where S is the unique set that satisfies $P \in \Delta^S$. Let $\Delta \equiv \prod_{S \in \mathfrak{S}} \Delta^S$. Each member in Δ is referred to as a *protocol*. Let $\precsim$ be an arbitrary total order on N. For $P \equiv (P^S)_{S \in \mathfrak{S}} \in \Delta$, $\delta \in (0,1)$ and $\pi \in [0,1]$, define an extensive form game $G(P, \delta, \pi)$, which includes Chatterjee *et al.*'s (1993) model and Okada's (1996) as special cases.

The structure of the game is inductively defined. In the game, there are countably infinite superrounds, which are classified by superstates, and in a superround, there are countably infinite rounds, which are classified by states. We call $S \in \mathfrak{S}$ a *superstate*, and (S, s) such that $S \in \mathfrak{S}$ and $s \in S^*$ a *state*. In the game, bargaining proceeds as follows:

(I) The game begins with the superround with superstate N.

(II) At a superround with superstate S, bargaining proceeds as follows:

  (i) The superround with S begins with the round with state $(S, 0)$.

  (ii) At a round with state (S, s), bargaining proceeds as follows:

    (1) A player $i \in S$ is selected as a proposer with probability $P^S_{s,i}$.

    (2) Player i proposes a pair of a coalition including i and a payoff distribution for the coalition, i.e., (C, x) such that $C \in 2^S \setminus \{\emptyset\}$, $C \ni i$, $(x_k)_{k \in C} \in \mathbb{R}^C_+$ and $\sum_{k \in C} x_k \leq v(C)$.

    (3) Each player in the proposed coalition C announces accepting or rejecting the

[3] Whereas Okada (1996) assumes the superadditivity, Chatterjee *et al.* (1993) do not.

[4] In this paper, for any nonempty sets A and B, let B^A denote the set of mappings from A to B.

proposal sequentially according to $\precsim$ until a player in C rejects the proposal or all players in C accept it.[5]

If a player rejects the proposal, the game goes to the next round with state (S, s'), where $s' \in S$ is the rejector.

If all players in C accept the proposal, the superround with S ends.

If the superround with S ends with agreement with C such that $S \setminus C \in \mathfrak{S}$, the game goes to the next superround with superstate $S \setminus C$ with probability π and the game ends with probability $1 - \pi$.

If the superround with S ends with agreement with C such that $S \setminus C \notin \mathfrak{S}$, the game ends.

We say that a proposal is *accepted* if every responder accepts it and *rejected* otherwise. Player i obtains a payoff of $\delta^{t-1} x_i$ if a proposal $\left(C, (x_k)_{k \in C}\right)$ such that $C \ni i$ is accepted at the t-th round and nothing otherwise. δ is the common discount factor.

The GSP model is a generalization of the FOP model and the RP model. Let $F \equiv \left(F^S\right)_{S \in \mathfrak{S}} \in \Delta$ denote a protocol such that for all $S \in \mathfrak{S}$, $F^S_{s,s} = 1$ for all $s \in S$ and $F^S_{0,i} = 1$, where $i \precsim j$ for all $j \in S$. Then, $G(F, \delta, 1)$ is equivalent to the FOP model.[6] Let $R \equiv \left(R^S\right)_{S \in \mathfrak{S}} \in \Delta$ denote a protocol such that for all $S \in \mathfrak{S}$, $R^S_{s,i} = \frac{1}{|S|}$ for all $i \in S$ for all $s \in S^*$. Then, $G(R, \delta, 1)$ is equivalent to the RP model.[7]

Moreover, the FOP model and the RP model are two extreme special cases in the following sense: Take any $S \in \mathfrak{S}$. Let $\bar{\Delta}^S \equiv \left\{P \in \Delta^S \mid \forall i \in S, \forall s \in S, P_{i,i} \geq P_{s,i}\right\}$. Under $P \in \bar{\Delta}^S$, each player is recognized as a proposer with a higher probability at rounds following her rejection than at any other round. Note that

$$\left\{(P_{s,s})_{s \in S} \mid P \in \bar{\Delta}^S\right\} = \left\{(P_{s,s})_{s \in S} \mid P \in \Delta^S \wedge \operatorname{tr} P \geq 1\right\} = \left\{(x_s)_{s \in S} \in (0,1]^S \mid \sum_{s \in S} x_s \geq 1\right\}.$$

$F^S \in \arg\max_{P \in \bar{\Delta}^S} \operatorname{tr} P$ for all $S \in \mathfrak{S}$. $R^S \in \arg\min_{P \in \bar{\Delta}^S} \operatorname{tr} P$ for all $S \in \mathfrak{S}$.

In this paper, we consider pure strategies. The equilibrium concept employed in the

[5] Even if the order is contingent on states, the results of this paper do not alter.
[6] $G(F, \delta, 0)$ is equivalent to Selten's (1981) model.
[7] $G(R, \delta, 0)$ is equivalent to Okada's (1993) model.

paper is the stationary subgame perfect equilibrium (SSPE), which is the subgame perfect equilibrium such that each player takes the same actions at all rounds with states containing the same superstate.

3 Efficiency

In this section, we consider the efficiency defined as follows:

Definition 1. Take any $P \in \Delta$, any $\delta \in (0,1)$ and any $\pi \in [0,1]$. SSPE σ of $G(P,\delta,\pi)$ is a *subgame efficient* if in σ, at any round with state (S,s), every equilibrium proposal is immediately accepted at the round (*no delay*) and every player proposes full coalition S (*full-coalition formation*).

This definition is based on Okada (1996). Needless to say, discounting and the superadditivity necessitate the no-delay property and the full-coalition-formation property for the subgame efficiency, respectively.

The following theorem provides an equivalent condition for the subgame efficiency in $G(P,\delta,\pi)$:

Theorem 1. *Take any $P \equiv \left(P^S\right)_{S \in \mathfrak{S}} \in \Delta$, any $\delta \in (0,1)$ and any $\pi \in [0,1]$. Then, there exists a subgame efficient SSPE of $G(P,\delta,\pi)$ if and only if for all $S, S' \in \mathfrak{S}$ such that $S \supset S'$,*

$$\frac{v(S)}{\delta \operatorname{tr} P^S + (1-\delta)} \geq \frac{v(S')}{\delta \operatorname{tr}_{S'} P^S + (1-\delta)}. \tag{1}$$

Proof. See Appendix A. Q.E.D.

Remark. (1) is $\frac{v(S)}{\delta|S|+(1-\delta)} \geq \frac{v(S')}{\delta|S'|+(1-\delta)}$ if $P = F$ and $v(S) \geq \frac{v(S')}{\delta|S'|/|S|+(1-\delta)}$ if $P = R$.

Remark. The equivalent condition for the subgame efficiency does not depend on recognition probability profiles $\left(\left(P^S_{0,k}\right)_{k \in S}\right)_{S \in \mathfrak{S}}$ at initial rounds and total order $\precsim$.

This theorem is intuitively proved (on the necessity) as follows: Take any subgame efficient SSPE. Consider any $S \in \mathfrak{S}$. For $i \in S$, let v_i be player i's equilibrium expected payoff at the round with state (S,i). Notice that δv_k is player k's threshold to accept or reject a proposal. Then, by the subgame efficiency,

$$v_i = P^S_{i,i}\left(v(S) - \sum_{k\in S\setminus\{i\}} \delta v_k\right) + \sum_{k\in S\setminus\{i\}} P^S_{i,k}\delta v_i = P^S_{i,i}\left(v(S) - \sum_{k\in S}\delta v_k\right) + \sum_{k\in S} P^S_{i,k}\delta v_i.$$

By $\sum_{k\in S} P^S_{i,k} = 1$,

$$v_i = P^S_{i,i}\left(v(S) - \delta\sum_{k\in S} v_k\right) + \delta v_i. \tag{2}$$

Sum the above equation over S. Then, $\sum_{k\in S} v_k = \frac{\operatorname{tr} P^S}{\delta \operatorname{tr} P^S + (1-\delta)} v(S)$. Substitute this equation into (2). Then,

$$v_i = \frac{P^S_{i,i}}{\delta \operatorname{tr} P^S + (1-\delta)} v(S). \tag{3}$$

This equation is interpreted as follows: Consider the round with state (S,i). Interpret δ as the probability that the game continues to the next rounds. Then, with probability δ, every player j obtains a proposing opportunity of $P^S_{j,j}$ by rejecting a proposal, and thus player j's bargaining power is proportional to $P^S_{j,j}$, which implies that player i's share of $v(S)$ is $\frac{P^S_{i,i}}{\sum_{k\in S} P^S_{k,k}} = \frac{P^S_{i,i}}{\operatorname{tr} P^S}$. On the other hand, with probability $1-\delta$, the game ends at this round, and thus player j's bargaining power is proportional to $P^S_{i,j}$, which implies that player i's share of $v(S)$ is $\frac{P^S_{i,i}}{\sum_{k\in S} P^S_{i,k}} = P^S_{i,i}$. Thus, player i's ex ante share of $v(S)$ is the harmonic mean of $\frac{P^S_{i,i}}{\operatorname{tr} P^S}$ and $P^S_{i,i}$ weighted by δ and $1-\delta$, i.e., $\left(\delta\left(\frac{P^S_{i,i}}{\operatorname{tr} P^S}\right)^{-1} + (1-\delta)\left(P^S_{i,i}\right)^{-1}\right)^{-1} = \frac{P^S_{i,i}}{\delta \operatorname{tr} P^S + (1-\delta)}$. Hence, (3) holds.

Take any $S' \in \mathfrak{S}$ such that $S' \subset S$. Take any $i \in S'$. Consider the round with state (S,i). By the subgame efficiency, player i proposes the full coalition S. Then, $v(S) - \sum_{k\in S\setminus\{i\}} \delta v_k \geq v(S') - \sum_{k\in S'\setminus\{i\}} \delta v_k$. Thus, $v(S) - \sum_{k\in S}\delta v_k \geq v(S') - \sum_{k\in S'}\delta v_k$. Substitute (3) into this inequality. Then, we have (1).

Here, we introduce two binary relations on Δ as follows:

Definition 2. Take any $P \equiv \left(P^S\right)_{S\in\mathfrak{S}}, \hat{P} \equiv \left(\hat{P}^S\right)_{S\in\mathfrak{S}} \in \Delta$. (i) P is *more diagonal* than $\hat{P}$

if for all $S \in \mathfrak{S}$, $\operatorname{tr} P^S \geq \operatorname{tr} \hat{P}^S$. (ii) P is *proportional* to $\hat{P}$ if for all $S \in \mathfrak{S}$, for all $s \in S$, $\frac{P^S_{s,s}}{\operatorname{tr} P^S} = \frac{P^S_{s,s}}{\operatorname{tr} \hat{P}^S}$.

Remark. For all $S \in \mathfrak{S}$, for all $s \in S$, $F^S_{s,s} = 1$ and $R^S_{s,s} = \frac{1}{|S|}$. Thus, F is more diagonal than and proportional to R.

Remark. (i) "Be more diagonal than" is reflexive and transitive, i.e., a preorder, but not antisymmetric and thus not a partial order. (ii) "Be proportional to" is reflexive, symmetric and transitive, i.e., an equivalence relation.

We say that P is *less diagonal* than $\hat{P}$ if $\hat{P}$ is more diagonal than P.

Next, we compare protocols in terms of the subgame efficiency. According to the following theorem, as long as P is in the same equivalence class in Δ by "be proportional to", the more diagonal P is, the more difficult it is for the subgame efficiency in $G(P, \delta, \pi)$ to be achieved.

Theorem 2. *Take any $P, \hat{P} \in \Delta$, any $\delta \in (0, 1)$ and $\pi \in [0, 1]$. Suppose that P is more diagonal than and proportional to $\hat{P}$. Then, if there exists a subgame efficient SSPE of $G(P, \delta, \pi)$, there exists a subgame efficient SSPE of $G\left(\hat{P}, \delta, \pi\right)$.*

Proof. See Appendix B. Q.E.D.

Remark. As shown above, F is more diagonal than and proportional to R. Thus, from Theorem 2, for any $\delta \in (0, 1)$, if there exists a subgame efficient SSPE of $G(F, \delta, \pi)$, there also exists that of $G(R, \delta, \pi)$.

Theorem 2 is intuitively explained as follows: Let σ (resp. $\hat{\sigma}$) be a subgame efficient SSPE of $G(P, \delta, \pi)$ (resp. $G\left(\hat{P}, \delta, \pi\right)$). Take any $S \in \mathfrak{S}$. For any $i \in S$, let v_i (resp. $\hat{v}_i$) be player i's expected payoff at the round with state (S, i) by σ (resp. $\hat{\sigma}$). Interpret δ as the probability that the game continues to the next rounds as above. Then, with probability δ, player i's share of $v(S)$ in σ (resp. $\hat{\sigma}$) is $\frac{P^S_{i,i}}{\operatorname{tr} P^S}$ (resp. $\frac{\hat{P}^S_{i,i}}{\operatorname{tr} \hat{P}^S}$). Since P is proportional to $\hat{P}$, $\frac{P^S_{i,i}}{\operatorname{tr} P^S} = \frac{\hat{P}^S_{i,i}}{\operatorname{tr} \hat{P}^S}$. On the other hand, with probability $1 - \delta$, player i's share of $v(S)$ in σ (resp. $\hat{\sigma}$) is $P^S_{i,i}$ (resp. $\hat{P}^S_{i,i}$). Since P is more diagonal than $\hat{P}$, $P^S_{i,i} \geq \hat{P}^S_{i,i}$. Therefore, $v_i \geq \hat{v}_i$. Notice that δv_i (resp. $\delta \hat{v}_i$) is player i's threshold to accept or reject a proposal in σ (resp. $\hat{\sigma}$). Then, $v_i \geq \hat{v}_i$ implies that the subcoalition formation or the delay is more likely to occur in σ than in $\hat{\sigma}$.

Next, using Theorem 1, we present a condition for the subgame efficiency when players are sufficiently patient. We say that *there exists a subgame efficient SSPE of* $G(P,1,\pi)$ if for some $\delta^* \in (0,1)$, for any $\delta \in (\delta^*,1)$, there exists a subgame efficient SSPE of $G(P,\delta,\pi)$.

Theorem 3. *Take any* $P \equiv (P^S)_{S\in\mathfrak{S}} \in \Delta$ *and any* $\pi \in [0,1]$*. Then, there exists a subgame efficient SSPE of* $G(P,1,\pi)$ *if and only if for all* $S, S' \in \mathfrak{S}$ *such that* $S \supset S'$,

$$\frac{v(S)}{\operatorname{tr} P^S} \geq \frac{v(S')}{\operatorname{tr}_{S'} P^S}. \tag{4}$$

Proof. See Appendix C. Q.E.D.

Remark. If $P^S_{s,s} = P^S_{s',s'}$ for all $s, s' \in S$, (4) is equivalent to $\frac{v(S)}{|S|} \geq \frac{v(S')}{|S'|}$. As shown above, for all $S \in \mathfrak{S}$, for all $s \in S$, $F^S_{s,s} = 1$ and $R^S_{s,s} = \frac{1}{|S|}$. Thus, an equivalent condition for the subgame efficiency in $G(F,1,\pi)$ or $G(R,1,\pi)$ is that for all $S, S' \in \mathfrak{S}$ such that $S \supset S'$, $\frac{v(S)}{|S|} \geq \frac{v(S')}{|S'|}$.

Remark. Needless to say, the equivalent condition for the subgame efficiency does not depend on recognition probability profiles $\left(\left(P^S_{0,k}\right)_{k\in S}\right)_{S\in\mathfrak{S}}$ at initial rounds and total order $\precsim$.

Remark. Take any $\pi \in [0,1]$. There exists $P \in \Delta$ such that there exists a subgame efficient SSPE of $G(P,1,\pi)$ if and only if (S, v^S) has a nonempty core for any $S \in \mathfrak{S}$, where $v^S : 2^S \to \mathbb{R}$ is defined as $v^S(T) = v(T)$ for all $T \in 2^S$.[8]

By letting δ go to 1 in Theorem 1, we obtain Theorem 3.

Theorem 3 implies the following theorem:

Theorem 4. *Take any* $P, \hat{P} \in \Delta$ *and any* $\pi \in [0,1]$*. Suppose that* P *is proportional to* $\hat{P}$*. Then, there exists a subgame efficient SSPE of* $G(P,1,\pi)$ *if and only if there exists a subgame efficient SSPE of* $G\left(\hat{P},1,\pi\right)$.

Proof. By Theorem 3. Q.E.D.

[8] Yan (2002) shows that in Okada's (1993) model with different recognition probabilities, any core allocation of the underlying characteristic function form game is implemented as an equilibrium payoff profile of a game with some recognition probability profile in the limit $\delta \to 1$.

Remark. Then, F is proportional to R. Thus, in the limit $\delta \to 1$, the subgame efficiency in $G(F, \delta, \pi)$ is equivalent to that in $G(R, \delta, \pi)$.

In contrast to Theorem 2, in the limit $\delta \to 1$, as long as P is in the same equivalence class in Δ by "be proportional to", the diagonality of P does not matter for the subgame efficiency.

Theorem 4 can be intuitively explained by the intuition of Theorem 2. We reuse the notations for the intuition of Theorem 2. Under the interpretation of δ as the probability that the game continues to the next rounds, in the limit $\delta \to 1$, the game continues with certainty. Then, player i's share of $v(S)$ in σ (resp. $\hat{\sigma}$) is $\frac{P^S_{i,i}}{\operatorname{tr} P^S}$ (resp. $\frac{\hat{P}^S_{i,i}}{\operatorname{tr} \hat{P}^S}$). Since P is proportional to $\hat{P}$, $\frac{P^S_{i,i}}{\operatorname{tr} P^S} = \frac{\hat{P}^S_{i,i}}{\operatorname{tr} \hat{P}^S}$. Therefore, $v_i = \hat{v}_i$, i.e., each player's threshold to accept or reject a proposal does not vary between σ and $\hat{\sigma}$. Thus, in the limit $\delta \to 1$, the subgame efficiency in the games with P is equivalent to that in the games with $\hat{P}$.

Finally, the following theorem compares protocols in different terms:

Theorem 5. *Take any $\pi \in [0,1]$. For all $P \in \Delta$ such that there exists a subgame efficient SSPE of $G(P, 1, \pi)$, there exists $\hat{P} \in \Delta$ with $\hat{P}$ which is less diagonal than P and not P such that there exists a subgame efficient SSPE of $G\left(\hat{P}, 1, \pi\right)$.*

Proof. See Appendix D. Q.E.D.

Theorem 5 follows Theorem 4. An implication of Theorem 5 is that when we design bargaining procedures, using less diagonal protocols is desirable for the subgame efficiency.

4 Conclusion

In this paper, the following results are shown: (i) For any $\delta \in (0,1)$, there exists a subgame efficient SSPE of $G(P, \delta, \pi)$ if and only if $\frac{v(S)}{\delta \operatorname{tr} P^S + (1-\delta)} \geq \frac{v(S')}{\delta \operatorname{tr}_{S'} P^S + (1-\delta)}$ for all $S, S' \in \mathfrak{S}$ such that $S \supset S'$ (Theorem 1). (ii) For any $\delta \in (0,1)$, if P is more diagonal than and proportional to $\hat{P}$, the subgame efficiency is more difficultly achieved in $G(P, \delta, \pi)$ than in $G\left(\hat{P}, \delta, \pi\right)$ (Theorem 2). (iii) In the limit $\delta \to 1$, there exists a subgame efficient SSPE of $G(P, \delta, \pi)$ if and only if $\frac{v(S)}{\operatorname{tr} P^S} \geq \frac{v(S')}{\operatorname{tr}_{S'} P^S}$ for all $S, S' \in \mathfrak{S}$ such that $S \supset S'$ (Theorem 3). (iv) In the limit

$\delta \to 1$, as long as P is proportional to $\hat{P}$, the subgame efficiency is achieved in $G(P, \delta, \pi)$ if and only if it is achieved in $G\left(\hat{P}, \delta, \pi\right)$ (Theorem 4). (v) In the limit $\delta \to 1$, for all $P \in \Delta$ such that there exists a subgame efficient SSPE of $G(p, \delta, \pi)$, there exists $\hat{P} \in \Delta$ with $\hat{P}$ which is less diagonal than P and not P such that there exists a subgame efficient SSPE of $G\left(\hat{P}, \delta, \pi\right)$ (Theorem 5).

Appendix

A Proof of Theorem 1

Proof. (Sufficiency) Suppose that (1) holds for all $S, S' \in \mathfrak{S}$ such that $S \supset S'$. Consider strategy profile σ such that for any round with state (S, s), (i) any player $i \in S$ proposes $\left(S, \left(y_k^i\right)_{k \in S}\right)$ with $y_i^i = v(S) - \sum_{k \in S \setminus \{i\}} \delta x_i$ and $y_k^i = \delta x_k$ for any $k \in S \setminus \{i\}$ and (ii) any player $i \in S$ accepts a proposal if and only if her share by the proposal is greater than or equal to δx_i, where $x_i = \frac{P_{i,k}^S}{\delta \operatorname{tr} P^S + (1-\delta)} v(S)$ for $i \in S$. Note that for $i \in S$, $y_i^i = \frac{\delta P_{i,i}^S + (1-\delta)}{\delta \operatorname{tr} P^S + (1-\delta)} v(S) \geq 0$ and $\sum_{k \in S} y_k^i = v(S)$. Note also that $y_i^i \geq \delta x_i$. Finally, note that in σ, actions at a round with state (S, s) are the same as at a round with state (S, s') for all $s, s' \in S$. Consider any superround with superstate S. For $i \in S$, let v_i be player i's expected payoff at the round with state (S, i) by σ. Obviously, in σ, every player offers an acceptable proposal at every round $(*)$. Thus, v_i is computed as $v_i = \sum_{k \in S} P_{i,k}^S y_i^k = P_{i,i}^S \left(v(S) - \sum_{k \in S \setminus \{i\}} \delta x_k\right) + \left(1 - P_{i,i}^S\right) \delta x_i = x_i$. Consider any round with (S, s). First, we show the unimprovability of responding actions in the round. By $v_i = x_i$, every player's responding actions in σ are unimprovable. Next, consider the unimprovability of proposing actions of any player $i \in S$ at the round. Consider any one deviation by player i such that she offers an acceptable proposal with an arbitrary coalition $S' \in 2^S$ with $S' \ni i$. By the deviation, player i obtains a payoff of $v(S') - \sum_{k \in S' \setminus \{i\}} \delta x_k$ at most. Thus, player i's gain from the deviation is at most

$$
\begin{aligned}
&\left(v(S') - \sum_{k \in S' \setminus \{i\}} \delta x_k\right) - y_i^i \\
&\quad = v(S') - \delta \sum_{k \in S'} x_k - v(S) + \delta \sum_{k \in S} x_k \\
&\quad = v(S') - \frac{\delta \operatorname{tr}_{S'} P^S}{\delta \operatorname{tr} P^S + (1-\delta)} v(S) - v(S) + \frac{\delta \operatorname{tr} P^S}{\delta \operatorname{tr} P^S + (1-\delta)} v(S) \\
&\quad = v(S') - \frac{\delta \operatorname{tr}_{S'} P^S + (1-\delta)}{\delta \operatorname{tr} P^S + (1-\delta)} v(S),
\end{aligned}
$$

which is less than or equal to 0, by (1) in the case that $v(S') > 0$ and obviously in the case that $v(S') = 0$. Consider any one deviation by player i such that she offers an unacceptable proposal. Let j be the rejector. Then, player i obtains an expected payoff of $\sum_{k\in S} P^S_{j,k} y^k_i$ by the deviation. Thus, player i's gain from the deviation is $\sum_{k\in S} P^S_{j,k} y^k_i - y^i_i = \sum_{k\in S\setminus\{i\}} P^S_{j,k} y^k_i - \left(1 - P^S_{j,i}\right) y^i_i = -\left(1 - P^S_{j,i}\right)\left(y^i_i - \delta x_i\right) \leq 0$. Therefore, player i's proposing actions in σ is unimprovable. To sum up, σ is an SPE by the one deviation principle. Obviously, σ is stationary. By $(*)$, σ involves no delay. Obviously, σ satisfies the full-coalition formation. Thus, σ is subgame efficient.

(Necessity) Suppose that there exists a subgame efficient SSPE σ of $G(p, \delta, \pi)$. Take any $S, S' \in \mathfrak{S}$ such that $S \supset S'$. For $i \in S$, let v_i be player i's expected payoff at the round with state (S, i) by σ. Then,

$$v_i = P^S_{i,i}\left(v(S) - \sum_{k\in S\setminus\{i\}} \delta v_k\right) + \left(1 - P^S_{i,i}\right)\delta v_i = P^S_{i,i}\left(v(S) - \delta\sum_{k\in S} v_k\right) + \delta v_i. \tag{5}$$

Sum the above equation over S. Then, $\sum_{k\in S} v_k = \left(\operatorname{tr} P^S\right)\left(v(S) - \delta\sum_{k\in S} v_k\right) + \delta\sum_{k\in S} v_k$. Thus, $\sum_{k\in S} v_k = \frac{\operatorname{tr} P^S v(S)}{\delta \operatorname{tr} P^S + (1-\delta)}$. Substitute this into (5). Then,

$$v_i = \frac{P^S_{i,i} v(S)}{\delta \operatorname{tr} P^S + (1-\delta)}. \tag{6}$$

Consider proposing actions of any player $i \in S'$ at the round with state (S, i). Since σ is subgame efficient, player i's payoff at her proposing node at the round is $v(S) - \sum_{k\in S\setminus\{i\}} \delta v_k$. This must be greater than or equal to $v(S') - \sum_{k\in S'\setminus\{i\}} \delta v_k$ since σ is an SPE. Thus, $v(S) - \sum_{k\in S} \delta v_k \geq v(S') - \sum_{k\in S'} \delta v_k$. By (6), we have $v(S) - \frac{\delta \operatorname{tr} P^S v(S)}{\delta \operatorname{tr} P^S + (1-\delta)} \geq v(S') - \frac{\delta \operatorname{tr}_{S'} P^S v(S)}{\delta \operatorname{tr} P^S + (1-\delta)}$. Hence, we obtain (1). Q.E.D.

B Proof of Theorem 2

Proof. Suppose that there exists a subgame efficient SSPE of $G(P, \delta, \pi)$. Take any $S, S' \in \mathfrak{S}$ such that $S \supset S'$. Let $D \equiv \frac{v(S)}{\delta \operatorname{tr} \hat{P}^S + (1-\delta)} - \frac{v(S')}{\delta \operatorname{tr}_{S'} \hat{P}^S + (1-\delta)}$. By the supposition that there exists a subgame efficient SSPE of $G(p, \delta, \pi)$, Theorem 1 yields $\frac{v(S)}{\delta \operatorname{tr} P^S + (1-\delta)} \geq \frac{v(S')}{\delta \operatorname{tr}_{S'} P^S + (1-\delta)}$. This inequality implies that

$$D \geq \frac{\delta \operatorname{tr} P^S + (1-\delta)}{\delta \operatorname{tr} \hat{P}^S + (1-\delta)} \frac{v(S')}{\delta \operatorname{tr}_{S'} P^S + (1-\delta)} - \frac{v(S')}{\delta \operatorname{tr}_{S'} \hat{P}^S + (1-\delta)} =: D'. \tag{7}$$

Thus,

$$\begin{aligned} D' &\propto \left\{\delta \operatorname{tr} P^S + (1-\delta)\right\} \left\{\delta \operatorname{tr}_{S'} \hat{P}^S + (1-\delta)\right\} - \left\{\delta \operatorname{tr} \hat{P}^S + (1-\delta)\right\} \left\{\delta \operatorname{tr}_{S'} P^S + (1-\delta)\right\} \\ &= \delta (1-\delta) \left(\operatorname{tr} P^S - \operatorname{tr} \hat{P}^S + \operatorname{tr}_{S'} \hat{P}^S - \operatorname{tr}_{S'} P^S\right) + \delta^2 \left(\operatorname{tr} P^S \operatorname{tr}_{S'} \hat{P}^S - \operatorname{tr} \hat{P}^S \operatorname{tr}_{S'} P^S\right) \end{aligned}$$

holds.[9] Therefore, by $\frac{P^S_{i,i}}{\operatorname{tr} P^S} = \frac{\hat{P}^S_{i,i}}{\operatorname{tr} \hat{P}^S}$ for all $i \in S$ and the transitivity of $\propto$, we obtain

$$D' \propto \operatorname{tr} P^S - \operatorname{tr} \hat{P}^S + \frac{\operatorname{tr} \hat{P}^S}{\operatorname{tr} P^S} \operatorname{tr}_{S'} P^S - \operatorname{tr}_{S'} P^S = \left(\operatorname{tr} P^S - \operatorname{tr} \hat{P}^S\right) \left(1 - \frac{\operatorname{tr}_{S'} P^S}{\operatorname{tr} P^S}\right).$$

Notice that $\operatorname{tr} P^S \geq \operatorname{tr} \hat{P}^S$ since P is more diagonal than $\hat{P}$. Then, $D' \geq 0$. Thus, by (7), $D \geq 0$. Hence, by Theorem 1, there exists a subgame efficient SSPE of $G\left(\hat{P}, \delta, \pi\right)$. Q.E.D.

C Proof of Theorem 3

Proof. (Sufficiency) Suppose that (4) holds for all $S, S' \in \mathfrak{S}$ such that $S \supset S'$. Take any $\delta \in (0,1)$. Take any $S, S' \in \mathfrak{S}$ such that $S \supset S'$. Let $D \equiv \frac{v(S)}{\delta \operatorname{tr} P^S + (1-\delta)} - \frac{v(S')}{\delta \operatorname{tr}_{S'} P^S + (1-\delta)}$. Then,

$$D = \frac{(1-\delta)(v(S) - v(S')) + \delta \left(v(S) \operatorname{tr}_{S'} P^S - v(S') \operatorname{tr} P^S\right)}{\{\delta \operatorname{tr} P^S + (1-\delta)\}\{\delta \operatorname{tr}_{S'} P^S + (1-\delta)\}} \geq 0$$

by the supposition that $\frac{v(S)}{\operatorname{tr} P^S} \geq \frac{v(S')}{\operatorname{tr}_{S'} P^S}$. Therefore, Theorem 1 implies that there exists a subgame efficient SSPE of $G(P, \delta, \pi)$. Since δ is arbitrary, (i) holds.

(Necessity) we show the contraposition of the statement. Suppose that $\frac{v(S)}{\operatorname{tr} P^S} < \frac{v(S')}{\operatorname{tr}_{S'} P^S}$ for some $S, S' \in \mathfrak{S}$ such that $S \supset S'$. By this supposition, $\hat{\delta} \equiv \frac{v(S) - v(S')}{v(S) - v(S') + v(S') \operatorname{tr} P^S - v(S) \operatorname{tr}_{S'} P^S} \in [0,1)$. Take any $\delta^* \in (0,1)$. Consider $\bar{\delta} \equiv \max\left\{\frac{\hat{\delta}+1}{2}, \frac{\delta^*+1}{2}\right\} \in (\delta^*, 1)$. Let $D \equiv \frac{v(S')}{\bar{\delta} \operatorname{tr}_{S'} P^S + \left(1-\bar{\delta}\right)} -$

[9] For any $x, y \in \mathbb{R}$, we write $x \propto y$ if $x = ay$ for some $a \in \mathbb{R}_{++}$. Note that $\propto$ is transitive and for any $x, y \in \mathbb{R}$, $x \propto y$ and $y \geq 0$ imply $x \geq 0$.

$\frac{v(S)}{\bar{\delta}\operatorname{tr}P^S+\left(1-\bar{\delta}\right)}$. Then,

$$D=\frac{-\left(v\left(S\right)-v\left(S'\right)\right)+\bar{\delta}\left(v\left(S\right)-v\left(S'\right)+v\left(S'\right)\operatorname{tr}P^S-v\left(S\right)\operatorname{tr}_{S'}P^S\right)}{\left\{\bar{\delta}\operatorname{tr}_{S'}P^S+\left(1-\bar{\delta}\right)\right\}\left\{\bar{\delta}\operatorname{tr}P^S+\left(1-\bar{\delta}\right)\right\}}.$$

Notice that $v\left(S\right)-v\left(S'\right)+v\left(S'\right)\operatorname{tr}P^S-v\left(S\right)\operatorname{tr}_{S'}P^S>0$ by the supposition that $\frac{v(S)}{\operatorname{tr}P^S}<\frac{v(S')}{\operatorname{tr}_{S'}P^S}$. Then, by $\bar{\delta}\geq\frac{\hat{\delta}+1}{2}>\hat{\delta}$,

$$D>\frac{-\left(v\left(S\right)-v\left(S'\right)\right)+\hat{\delta}\left(v\left(S\right)-v\left(S'\right)+v\left(S'\right)\operatorname{tr}P^S-v\left(S\right)\operatorname{tr}_{S'}P^S\right)}{\left\{\bar{\delta}\operatorname{tr}_{S'}P^S+\left(1-\bar{\delta}\right)\right\}\left\{\bar{\delta}\operatorname{tr}P^S+\left(1-\bar{\delta}\right)\right\}}.$$

Thus, by the definition of $\hat{\delta}$, we obtain $D>0$. Therefore, Theorem 1 implies that there exists no subgame efficient SSPE of $G\left(P,\bar{\delta},\pi\right)$. Q.E.D.

D Proof of Theorem 5

Proof. Take any $P\equiv\left(P^S\right)_{S\in\mathfrak{S}}\in\Delta$ such that there exists a subgame efficient SSPE of $G\left(P,1,\pi\right)$. Take a $\hat{P}\equiv\left(\hat{P}^S\right)_{S\in\mathfrak{S}}\in\Delta$ such that $\hat{P}^S_{s,s}=\frac{1}{2}P^S_{s,s}$ for all $S\in\mathfrak{S}$ and all $s\in S$. Note that there exists such a $\hat{p}$ in P since $\hat{P}^S_{s,s}\in(0,1]$ by $P^S_{s,s}\in(0,1]$ for all $S\in\mathfrak{S}$ and all $s\in S$. Obviously, $\hat{P}$ is less diagonal than P and not P. Obviously, $\hat{P}$ is proportional to P. Thus, by Theorem 4, the conclusion of Theorem 5 is obtained. Q.E.D.

References

[1] Chatterjee, K., B. Dutta, D. Ray and K. Sengupta, 1993, "A noncooperative theory of coalitional bargaining," *Review of Economic Studies* 60, 463-477.

[2] Okada, A., 1993, "A noncooperative coalitional bargaining game with random proposers," working paper, Kyoto University.

[3] Okada, A., 1996, "A noncooperative coalitional bargaining game with random proposers," *Games and Economic Behavior* 16, 97-108.

[4] Selten, R., 1981, "A noncooperative model of characteristic-function bargaining," in: R.J. Aumann *et al.*, eds., *Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern*, 131-151.

[5] Yan, H., 2002, "Noncooperative selection of the core," *International Journal of Game Theory* 31, 527-540.