Outcome of Majority Voting in Multiple Undesirable Facility Location Problems

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Abstract

We consider the outcome of majority voting in multiple undesirable facility location problems where the locations of two facilities are planned, any individual is concerned about the location of the nearest facility but not about the location of the other facility, and any individual prefers that the location of the nearest facility be as far as possible from his/her location. We show that a Condorcet winner is a subset of the set of pendant vertices and the vertices adjacent to pendant vertices on a tree network with an odd number of individuals. Furthermore, we derive a necessary and sufficient condition for a set of locations to be a Condorcet winner on a line network with an odd number of individuals.

Keywords: Locations of multiple undesirable facilities; Majority voting; Condorcet winner

JEL Classification: D72; R53; H41

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1 Introduction

The locations of facilities have an important influence on the daily life of individuals. For example, if a park is located in the vicinity of an individual, he/she surely enjoys beautiful scenery every day. Conversely, if a dump is located in the vicinity of an individual, he/she is surely bothered by bad odor every day. Hence, individuals are very interested in the locations of facilities. Then, if individuals collectively choose the locations of facilities, which locations do they choose? Furthermore, are the chosen locations socially desirable? Many studies have been devoted to answering these questions in cases where individuals collectively choose the locations of facilities through majority voting by employing a Condorcet winner, which is unbeatable through pairwise majority voting, as a solution under majority voting.[1,2]

Initially, the studies focused on single desirable facility location problems where the location of a single facility is planned, and any individual prefers that the location of the facility be as close as possible to his/her location. In these problems, a Condorcet winner is a location that is unbeatable through pairwise majority voting. Hansen and Thisse (1981) showed that the set of Condorcet winners equals the set of medians on a tree network.[3] Labbe (1985) showed that the set of Condorcet winners equals either the empty set or the set of medians on a cactus network. Furthermore, Hansen and Labbé (1988) derived an algorithm for finding a Condorcet winner on a general network. In these problems, a Condorcet winner is not necessarily a median on a general network.[4] However, Hansen and Thisse (1981) showed that the ratio of the average distance from an individual's location to a Condorcet winner to the average distance from an individual's location to a median is bounded above on a general network.

Subsequently, the studies began to focus on single undesirable facility location problems where the location of a single facility is planned, and any individual prefers that the location of the facility be as far as possible from his/her location. In these problems, a Condorcet winner is a location that is unbeatable through pairwise majority voting. Labbé (1990) showed that a Condorcet winner is a pendant vertex or a bottleneck location on a general network with an odd number of individuals. Furthermore, the study revealed that the set of Condorcet winners equals the set of the pendant vertices that satisfy some condition on a tree network with an

[1] Some studies have been devoted to answering those questions in cases where individuals collectively choose the location of a facility through unanimity bargaining by employing the equilibrium location in alternating-offer bargaining as a solution under unanimity bargaining. For example, see Kawamori and Yamaguchi (2010) for single desirable facility location problems.

[2] Since the existence of a Condorcet solution is not assured, other voting solutions weaker than a Condorcet solution have been also employed. For example, see Bandelt and Labbé (1986) for a Simpson solution, Campos and Moreno Pérez (2000) for a tolerant Condorcet solution, and Campos Rodríguez and Moreno Pérez (2003) for their mixture.

[3] A p-median is a set of p locations such that the average distance from an individual's location to the nearest location in a set of p locations is minimized, that is, a Benthamite social welfare maximizer in p-desirable facility location problems.

[4] Bandelt (1985) characterized the networks on which the set of Condorcet winners equals the set of medians.

odd number of individuals. In these problems, a Condorcet winner is not necessarily an antimedian, even on a line network.[5] However, Labbé (1990) showed that the ratio of the average distance from an individual's location to an antimedian to the average distance from an individual's location to a Condorcet winner is bounded above on a general network.

Recently, the studies addressed multiple desirable facility location problems where the locations of multiple facilities are planned, any individual is concerned about the location of the nearest facility but not about the locations of the other facilities, and any individual prefers that the location of the nearest facility be as close as possible to his/her location. In these problems, a Condorcet winner is a set of locations that is unbeatable through pairwise majority voting. Barberà and Beviá (2006) showed that a Condorcet winner is efficient, internally consistent, and Nash stable on a line network. Hajduková (2010) derived the additional necessary conditions and a sufficient condition for a set of locations to be a Condorcet winner on a line network. Furthermore, Campos Rodríguez and Moreno Pérez (2008) derived an algorithm for finding a Condorcet winner on a general network. In these problems, a Condorcet winner is not necessarily a median, even on a line network. Furthermore, the ratio of the average distance from an individual's location to the nearest location in a Condorcet winner to the average distance from an individual's location to the nearest location in a median is not bounded above, even on a line network.

Following these studies, this study is devoted to multiple undesirable facility location problems where the locations of multiple facilities are planned, any individual is concerned about the location of the nearest facility but not about the locations of the other facilities, and any individual prefers that the location of the nearest facility be as far as possible from his/her location. In these problems, a Condorcet winner is a set of locations that is unbeatable through pairwise majority voting. We assume that the locations of two facilities are planned. We show that a Condorcet winner is a subset of the set of pendant vertices and the vertices adjacent to pendant vertices on a tree network with an odd number of individuals. Furthermore, we derive a necessary and sufficient condition for a set of locations to be a Condorcet winner on a line network with an odd number of individuals. In these problems, we show that the ratio of the average distance from an individual's location to the nearest location in an antimedian to the average distance from an individual's location to the nearest location in a Condorcet winner is not bounded above, even on a line network.

The remainder of this paper is organized as follows. Section 2 describes our model, and Section 3 presents

[5] A p-antimedian is a set of p locations such that the average distance from an individual's location to the nearest location in a set of p locations is maximized, that is, a Benthamite social welfare maximizer in p-undesirable facility location problems.

our result.

2 Model

In this section, we describe our model.

Let (V, E) be a geometric graph. For some metric space X, V is a finite subset of X, and E is a finite set of some continuous injections from $[0,1]$ to X such that for any $e \in E$, $e(\{0,1\}) \subset V$ and $e([0,1] \setminus \{0,1\}) \subset X \setminus V$, and for any $(e, e') \in E^2$ with $e \neq e'$, $e((0,1)) \cap e'((0,1)) = \emptyset$. We call $v \in V$ a vertex, and for any $e \in E$, $e([0,1])$, an edge. We assume that $|V| \geq 3$. Let $N = \left(\bigcup_{e \in E} e([0,1])\right) \cup \left(\bigcup_{v \in V} \{v\}\right)$. We call N a network, and $n \in N$, a location. For any $(n, n') \in N^2$, we call $R \in 2^N$ a route between n and n' if (i) $R \ni n$, $R \ni n'$, and R is connected and (ii) there does not exist $S \subsetneq R$ such that $S \ni n$, $S \ni n'$, and S is connected. We assume that there is a unique route between two locations of the network; in other words, the network is a tree. For any $(n, n') \in N^2$, let $R(n, n')$ be the route between n and n'. Let d be a map from N^2 to $\mathbb{R}_+$ such that for any $(n, n') \in N^2$, $d(n, n')$ denotes the length of $R(n, n')$. Then, d denotes a metric on N.

Let I be a nonempty finite set of individuals. We assume that $|I|$ is odd. Individual $i \in I$ is located at a vertex $v_i \in V$ of the network. For any $i \in I$, let d_i be a map from N to $\mathbb{R}_+$ such that for any $n \in N$, $d_i(n) = d(n, v_i)$. Then, $d_i(n)$ denotes individual i's distance from location n.

These individuals collectively choose the locations of two undesirable facilities on the network through majority voting. It is permissible for the facilities to be located only on the vertices of the network, but not on the same vertex of the network. Let $\mathcal{L} = \{L \subset V \| L| = 2\}$. Then, $\{\ell, \ell'\} \in \mathcal{L}$ denotes a set of locations that is a candidate for the set of the locations of the facilities. Any individual is concerned about the location of the nearest facility but not about the location of the other facility. Any individual prefers that the location of the nearest facility be as far as possible from his/her location. For any $i \in I$, let D_i be a map from $\mathcal{L}$ to $\mathbb{R}_+$ such that for any $\{\ell, \ell'\} \in \mathcal{L}$, $D_i(\{\ell, \ell'\}) = \min\{d_i(\ell), d_i(\ell')\}$. Then, $D_i(\{\ell, \ell'\})$ denotes individual i's distance from the location of the nearest facility when $\{\ell, \ell'\}$ is the set of the locations of the facilities. We employ a Condorcet winner, which is unbeatable through pairwise majority voting, as a solution under majority voting. In our model, a Condorcet winner is a set of locations that is unbeatable through pairwise majority voting.

Definition 1. $C \in \mathcal{L}$ is a Condorcet winner if for any $L \in \mathcal{L}$,

$$|\{i \in I | D_i(C) < D_i(L)\}| \leq \frac{|I|}{2}.$$

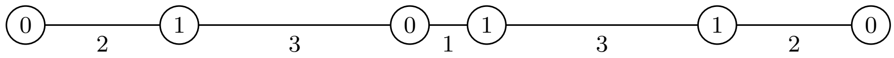

Figure 1: An example where a Condorcet winner does not exist: the number in a vertex denotes the number of the individuals located on the vertex; the number under an edge denotes the length of the edge

Let $\mathcal{C}$ denote the set of Condorcet winners.

3 Result

In this section, we present the result.

First, we derive a necessary condition for a set of locations to be a Condorcet winner on a tree network. We refer to a vertex of degree one as a pendant vertex. Let P be the set of pendant vertices. Let Q be the set of the vertices adjacent to the pendant vertices.

Proposition 1. *Suppose that $C \in \mathcal{C}$. Then, $C \subset P \cup Q$.*

Proof. Suppose that $\{c, c'\} \in \mathcal{C}$. Suppose that $c \notin P \cup Q$. Let $V_1 = \{c\}$, $V_2 = \{v \in V | d(v, c') = d(v, c) + d(c, c')\} - V_1$, and $V_3 = V - V_1 - V_2$. Then, $\{V_1, V_2, V_3\}$ is a partition of V. Note that since $c \notin P \cup Q$, $|V_2| \geq 2$ and $|V_3| \geq 2$. For any $j \in \{1, 2, 3\}$, let $I_j = \{i \in I | v_i \in V_j\}$. Then, $\{I_1, I_2, I_3\}$ is a partition of I. Note that since $|I|$ is odd, either $|I_1 \cup I_2| > \frac{|I|}{2}$ or $|I_1 \cup I_3| > \frac{|I|}{2}$. Suppose that $|I_1 \cup I_2| > \frac{|I|}{2}$. Note that for any $L \in \mathcal{L}$ such that $L \subset V_3$, for any $i \in I_1 \cup I_2$, $D_i(\{c, c'\}) < D_i(L)$. Since $\{c, c'\} \in \mathcal{C}$, this is a contradiction. Suppose that $|I_1 \cup I_3| > \frac{|I|}{2}$. Note that for any $L \in \mathcal{L}$ such that $L \subset V_2$, for any $i \in I_1 \cup I_3$, $D_i(\{c, c'\}) < D_i(L)$. Since $\{c, c'\} \in \mathcal{C}$, this is a contradiction. Q.E.D.

By Proposition 1, it is sufficient for finding a Condorcet winner to consider only a subset of the set of pendant vertices and the vertices adjacent to pendant vertices.

Unfortunately, a Condorcet winner does not necessarily exist, even on a line network. An example is shown in Figure 1. As shown by Hansen and Thisse (1981) and Labbé (1990), in single facility location problems, a Condorcet winner exists on a tree network. However, as shown by Barberà and Beviá (2006) and above, in multiple facility location problems, a Condorcet winner does not necessarily exist, even on a line network. Hence, in multiple facility location problems, other voting solutions weaker than a Condorcet solution may be needed for considering the full outcome of majority voting. However, if a Condorcet winner exists on a line network, we can find it easily.

Hereafter, we assume that the network is a line. Let m be a map from N^2 to N such that for any $(n, n') \in N^2$, $m(n, n') \in R(n, n')$ and $d(n, m(n, n')) = d(n', m(n, n'))$. Then, $m(n, n')$ denotes the middle location between locations n and n'. Let p and p' denote the pendant vertices. Let q and q' denote the vertices adjacent to pendant vertices p and p' respectively. Then, by Proposition 1, only $\{p, q\}$, $\{q', p'\}$, $\{p, q'\}$, $\{q, p'\}$, $\{p, p'\}$, and $\{q, q'\}$ are candidates that can be a Condorcet winner. For any set of locations, we derive a necessary and sufficient condition for the set of locations to be a Condorcet winner.

Proposition 2. *$\{p, q\} \in \mathcal{C}$ if and only if the following conditions are met: (i) $|\{i \in I | v_i \in R(m(q, q'), p')\}| > \frac{|I|}{2}$; and (ii) $|\{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}| > \frac{|I|}{2}$.*

Proof. (Necessity) Suppose that $\{p, q\} \in \mathcal{C}$. (i) Suppose that $|\{i \in I | v_i \in R(m(q, q'), p')\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, $|I - \{i \in I | v_i \in R(m(q, q'), p')\}| > \frac{|I|}{2}$. Note that for any $i \in I - \{i \in I | v_i \in R(m(q, q'), p')\}$, $D_i(\{p, q\}) < D_i(\{q', p'\})$. Since $\{p, q\} \in \mathcal{C}$, this is a contradiction. (ii) Suppose that $|\{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, $|I - \{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}| > \frac{|I|}{2}$. Note that for any $i \in I - \{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}$, $D_i(\{p, q\}) < D_i(\{p, p'\})$. Since $\{p, q\} \in \mathcal{C}$, this is a contradiction.

(Sufficiency) Suppose that (i) $|\{i \in I | v_i \in R(m(q, q'), p')\}| > \frac{|I|}{2}$ and (ii) $|\{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}| > \frac{|I|}{2}$. Then, $|\{i \in I | v_i \in R(q, p')\}| > \frac{|I|}{2}$. Note that for any $L \in \mathcal{L}$ such that $L \cap \{p\} \neq \emptyset$ and $L \cap \{q\} = \emptyset$, for any $i \in \{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}$, $D_i(\{p, q\}) \geq D_i(L)$. Note that for any $L \in \mathcal{L}$ such that $L \cap \{p\} = \emptyset$ and $L \cap \{q\} \neq \emptyset$, for any $i \in \{i \in I | v_i \in R(q, p')\}$, $D_i(\{p, q\}) \geq D_i(L)$. Note that for any $L \in \mathcal{L}$ such that $L \cap \{p\} = \emptyset$ and $L \cap \{q\} = \emptyset$, for any $i \in \{i \in I | v_i \in R(m(q, q'), p')\}$, $D_i(\{p, q\}) \geq D_i(L)$. Therefore, $\{p, q\} \in \mathcal{C}$. Q.E.D.

Proposition 3. *$\{q', p'\} \in \mathcal{C}$ if and only if the following conditions are met: (i) $|\{i \in I | v_i \in R(p, m(q, q'))\}| > \frac{|I|}{2}$; and (ii) $|\{i \in I | v_i \in R(p, m(p, q')) \cup \{p'\}\}| > \frac{|I|}{2}$.*

Proof. Since the proof of this proposition is analogous to that of Proposition 2, we omit the proof. Q.E.D.

Proposition 4. *$\{p, q'\} \in \mathcal{C}$ if and only if the following conditions are met: (i) for any adjacent vertices $v, v' \in V$ such that $d(p, v) < d(p, v')$ and $\{v, v'\} \cap \{q', p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p, v), m(v', q'))\}| > \frac{|I|}{2}$; and (ii) for any $v \in V$ such that $\{v\} \cap \{p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p, v), m(v, q')) \cup \{p'\}\}| > \frac{|I|}{2}$.*

Proof. (Necessity) Suppose that $\{p, q'\} \in \mathcal{C}$. (i) Suppose that for some adjacent vertices $v, v' \in V$ such that $d(p, v) < d(p, v')$ and $\{v, v'\} \cap \{q', p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p, v), m(v', q'))\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, $|I - \{i \in I | v_i \in R(m(p, v), m(v', q'))\}| > \frac{|I|}{2}$. Note that for any $i \in I - \{i \in I | v_i \in R(m(p, v), m(v', q'))\}$, $D_i(\{p, q'\}) < D_i(\{v, v'\})$. Since $\{p, q'\} \in \mathcal{C}$, this is a contradiction. (ii) Suppose that for some $v \in V$

such that $\{v\} \cap \{p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p,v), m(v,q')) \cup \{p'\}\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, $|I - \{i \in I | v_i \in R(m(p,v), m(v,q')) \cup \{p'\}\}| > \frac{|I|}{2}$. Note that for any $i \in I - \{i \in I | v_i \in R(m(p,v), m(v,q')) \cup \{p'\}\}$, $D_i(\{p,q'\}) < D_i(\{v,p'\})$. Since $\{p,q'\} \in \mathcal{C}$, this is a contradiction.

(Sufficiency) Suppose that (i) for any adjacent vertices $v, v' \in V$ such that $d(p,v) < d(p,v')$ and $\{v,v'\} \cap \{q',p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p,v), m(v',q'))\}| > \frac{|I|}{2}$, and (ii) for any $v \in V$ such that $\{v\} \cap \{p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p,v), m(v,q')) \cup \{p'\}\}| > \frac{|I|}{2}$. Then, for any vertices $v, v' \in V$ such that $d(p,v) < d(p,v')$ and $\{v,v'\} \cap \{q',p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p,v), m(v',q'))\}| > \frac{|I|}{2}$. Note that for any $\{v,v'\} \in \mathcal{L}$ such that $v' = p'$, for any $i \in \{i \in I | v_i \in R(m(p,v), m(v,q')) \cup \{p'\}\}$, $D_i(\{p,q'\}) \geq D_i(\{v,v'\})$. Note that for any $\{v,v'\} \in \mathcal{L}$ such that $v \neq p'$ and $v' = q'$, for any $i \in \{i \in I | v_i \in R(m(p,v), q') \cup \{p'\}\}$, $D_i(\{p,q'\}) \geq D_i(\{v,v'\})$. Note that for any $\{v,v'\} \in \mathcal{L}$ such that $d(p,v) < d(p,v')$ and $\{v,v'\} \cap \{q',p'\} = \emptyset$, for any $i \in \{i \in I | v_i \in R(m(p,v), m(v',q'))\}$, $D_i(\{p,q'\}) \geq D_i(\{v,v'\})$. Therefore, $\{p,q'\} \in \mathcal{C}$. Q.E.D.

Proposition 5. *$\{q,p'\} \in \mathcal{C}$ if and only if the following conditions are met: (i) for any adjacent vertices $v, v' \in V$ such that $d(p,v) < d(p,v')$ and $\{v,v'\} \cap \{p,q\} = \emptyset$, $|\{i \in I | v_i \in R(m(q,v), m(v',p'))\}| > \frac{|I|}{2}$; and (ii) for any $v \in V$ such that $\{v\} \cap \{p\} = \emptyset$, $|\{i \in I | v_i \in \{p\} \cup R(m(q,v), m(v,p'))\}| > \frac{|I|}{2}$.*

Proof. Since the proof of this proposition is analogous to that of Proposition 4, we omit the proof. Q.E.D.

Proposition 6. *$\{p,p'\} \in \mathcal{C}$ if and only if the following conditions are met: for any adjacent vertices $v, v' \in V$ such that $d(p,v) < d(p,v')$, $|\{i \in I | v_i \in R(m(p,v), m(v',p'))\}| > \frac{|I|}{2}$.*

Proof. (Necessity) Suppose that $\{p,p'\} \in \mathcal{C}$. Suppose that for some adjacent vertices $v, v' \in V$ such that $d(p,v) < d(p,v')$, $|\{i \in I | v_i \in R(m(p,v), m(v',p'))\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, $|I - \{i \in I | v_i \in R(m(p,v), m(v',p'))\}| > \frac{|I|}{2}$. Note that for any $i \in I - \{i \in I | v_i \in R(m(p,v), m(v',p'))\}$, $D_i(\{p,p'\}) < D_i(\{v,v'\})$. Since $\{p,p'\} \in \mathcal{C}$, this is a contradiction.

(Sufficiency) Suppose that for any adjacent vertices $v, v' \in V$ such that $d(p,v) < d(p,v')$, $|\{i \in I | v_i \in R(m(p,v), m(v',p'))\}| > \frac{|I|}{2}$. Then, for any vertices $v, v' \in V$ such that $d(p,v) < d(p,v')$, $|\{i \in I | v_i \in R(m(p,v), m(v',p'))\}| > \frac{|I|}{2}$. Note that for any $\{v,v'\} \in \mathcal{L}$ such that $d(p,v) < d(p,v')$, for any $i \in \{i \in I | \in R(m(p,v), m(v',p'))\}$, $D_i(\{p,p'\}) \geq D_i(\{v,v'\})$. Therefore, $\{p,p'\} \in \mathcal{C}$. Q.E.D.

Proposition 7. *$\{q,q'\} \in \mathcal{C}$ if and only if the following conditions are met: (i) $|V| = 5$; (ii) $m(q,q') \in V$; (iii) $|\{i \in I | v_i \in \{p, m(q,q')\}\}| > \frac{|I|}{2}$; (iv) $|\{i \in I | v_i \in \{m(q,q'), p'\}\}| > \frac{|I|}{2}$; and (v) $|\{i \in I | v_i \in \{p,p'\}\}| > \frac{|I|}{2}$.*

Proof. (Necessity) Suppose that $\{q, q'\} \in \mathcal{C}$. (i) Suppose that $|V| = 4$. Since $|I|$ is odd, either $|\{i \in I | v_i \in \{p, q\}\}| > \frac{|I|}{2}$ or $|\{i \in I | v_i \in \{q', p'\}\}| > \frac{|I|}{2}$. Suppose that $|\{i \in I | v_i \in \{p, q\}\}| > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in \{p, q\}\}$, $D_i(\{q, q'\}) < D_i(\{q', p'\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. Suppose that $|\{i \in I | v_i \in \{q', p'\}\}| > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in \{q', p'\}\}$, $D_i(\{q, q'\}) < D_i(\{p, q\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. Suppose that $|V| \geq 6$. Since $|I|$ is odd, either $|\{i \in I | v_i \in R(q, q')\}| > \frac{|I|}{2}$ or $|\{i \in I | v_i \in \{p, p'\}\}| > \frac{|I|}{2}$. Suppose that $|\{i \in I | v_i \in R(q, q')\}| > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in R(q, q')\}$, $D_i(\{q, q'\}) < D_i(\{p, p'\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. Suppose that $|\{i \in I | v_i \in \{p, p'\}\}| > \frac{|I|}{2}$. Note that for any $L \in \mathcal{L}$ such that $L \cap \{p, q, q', p'\} = \emptyset$, for any $i \in \{i \in I | v_i \in \{p, p'\}\}$, $D_i(\{q, q'\}) < D_i(L)$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. (ii) Suppose that $m(q, q') \notin V$. Since $|I|$ is odd, either $|\{i \in I | v_i \in R(p, m(q, q')) \setminus \{m(q, q')\}\}| > \frac{|I|}{2}$ or $|\{i \in I | v_i \in R(m(q, q'), p') \setminus \{m(q, q')\}\}| > \frac{|I|}{2}$. Suppose that $|\{i \in I | v_i \in R(p, m(q, q')) \setminus \{m(q, q')\}\}| > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in R(p, m(q, q')) \setminus \{m(q, q')\}\}$, $D_i(\{q, q'\}) < D_i(\{q', p'\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. Suppose that $|\{i \in I | v_i \in R(m(q, q'), p') \setminus \{m(q, q')\}\}| > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in R(m(q, q'), p') \setminus \{m(q, q')\}\}$, $D_i(\{q, q'\}) < D_i(\{p, q\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. (iii) Suppose that $|\{i \in I | v_i \in \{p, m(q, q')\}\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, $\{i \in I | v_i \in \{q, q', p'\}\} > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in \{q, q', p'\}\}$, $D_i(\{q, q'\}) < D_i(\{p, m(q, q')\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. (iv) Suppose that $|\{i \in I | v_i \in \{m(q, q'), p'\}\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, $\{i \in I | v_i \in \{p, q, q'\}\} > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in \{p, q, q'\}\}$, $D_i(\{q, q'\}) < D_i(\{m(q, q'), p'\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. (v) Suppose that $|\{i \in I | v_i \in \{p, p'\}\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, $\{i \in I | v_i \in \{q, m(q, q'), q'\}\} > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in \{q, m(q, q'), q'\}\}$, $D_i(\{q, q'\}) < D_i(\{p, p'\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction.

(Sufficiency) Suppose that (i) $|V| = 5$, (ii) $m(q, q') \in V$, (iii) $|\{i \in I | v_i \in \{p, m(q, q')\}\}| > \frac{|I|}{2}$, (iv) $|\{i \in I | v_i \in \{m(q, q'), p'\}\}| > \frac{|I|}{2}$, and (v) $|\{i \in I | v_i \in \{p, p'\}\}| > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in \{p, q, m(q, q')\}\}$, $D_i(\{q, q'\}) \geq D_i(\{p, q\})$. Note that for any $i \in \{i \in I | v_i \in \{p, m(q, q')\}\}$, $D_i(\{q, q'\}) \geq D_i(\{p, m(q, q')\})$. Note that for any $i \in \{i \in I | v_i \in \{p, m(q, q'), q', p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{p, q'\})$. Note that for any $i \in \{i \in I | v_i \in \{p, p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{p, p'\})$. Note that for any $i \in \{i \in I | v_i \in \{p, q, m(q, q')\}\}$, $D_i(\{q, q'\}) \geq D_i(\{q, m(q, q')\})$. Note that for any $i \in \{i \in I | v_i \in \{p, q, m(q, q'), p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{q, p'\})$. Note that for any $i \in \{i \in I | v_i \in \{m\{q, q'\}, q', p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{m(q, q'), q'\})$. Note that for any $i \in \{i \in I | v_i \in \{m(q, q'), p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{m(q, q'), p'\})$. Note that for any $i \in \{i \in I | v_i \in \{m(q, q'), q', p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{q', p'\})$. Therefore, $\{q, q'\} \in \mathcal{C}$. Q.E.D.

By Propositions 1–7, if a Condorcet winner exists on a line network, we can find it easily.

Finally, we evaluate a Condorcet winner according to the Benthamite criterion. $M \in \mathcal{L}$ is an antimedian if

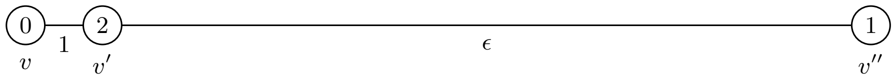

Figure 2: An example where the set of Condorcet winners does not intersect with the set of antimedians: ϵ denotes a number greater than 2; the number in a vertex denotes the number of the individuals located on the vertex; the number under an edge denotes the length of the edge

for any $L \in \mathcal{L}$, $\sum_{i \in I} D_i(M) \geq \sum_{i \in I} D_i(L)$. That is, an antimedian is a Benthamite social welfare maximizer. Unfortunately, a Condorcet winner is not necessarily an antimedian. Furthermore, the ratio of the average distance from an individual's location to the nearest location in an antimedian to the average distance from an individual's location to the nearest location in a Condorcet winner is not bounded above. An example is shown in Figure 2. In this example, the unique Condorcet winner $\{v, v''\}$ is not the unique antimedian $\{v, v'\}$. Furthermore, the ratio $\frac{\sum_{i \in I} D_i(\{v,v'\})/|I|}{\sum_{i \in I} D_i(\{v,v''\})/|I|} = \frac{\epsilon}{2}$ of the average distance from an individual's location to the nearest location in the unique antimedian to the average distance from an individual's location to the nearest location in the unique Condorcet winner goes to infinity as ϵ goes to infinity. As shown by Hansen and Thisse (1981), in single desirable facility location problems, the ratio of the average distance from an individual's location to a Condorcet winner to the average distance from an individual's location to a median is bounded above on a general network, and the ratio is 1 on a tree network. Furthermore, as shown by Labbé (1990), in single undesirable facility location problems, the ratio of the average distance from an individual's location to an antimedian to the average distance from an individual's location to a Condorcet winner is bounded above on a general network. However, in multiple desirable facility location problems, the ratio of the average distance from an individual's location to the nearest location in a Condorcet winner to the average distance from an individual's location to the nearest location in a median is not bounded above, even on a line network. Furthermore, as shown above, in multiple undesirable facility location problems, the ratio of the average distance from an individual's location to the nearest location in an antimedian to the average distance from an individual's location to the nearest location in a Condorcet winner is not bounded above, even on a line network. Hence, in multiple facility location problems, mechanisms other than majority voting may be needed for implementing socially desirable outcomes from the viewpoint of the Benthamite criterion.

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