# Asymmetric Equilibria under Price-cap Regulation

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**Abstract**

We discuss Cournot and Stackelberg duopoly models where the firms are regulated by a price-cap regulation. The symmetric and asymmetric Cournot equilibria under the price-cap regulation are characterized. Moreover, we show that a unique Stackelberg equilibrium exists and relate that to the Cournot equilibria. We present several comparative statics results with respect to the outcomes of the Cournot and Stackelberg models. Moreover, we consider an endogenous timing duopoly game where the firms engage in Stackelberg or Cournot competition depending on the pair of their actions in a preplay stage. Finally, we consider the welfare effect of a change in a price-cap level. When asymmetric equilibria are focused on, a reduction in a price-cap level can be socially harmful even if the price-cap level is more than the competitive price.

**Keywords**: Price-cap regulations; Cournot competition; Stackelberg competition; Asymmetric equilibria; Endogenous role

**JEL classification codes**: **D43; L11; L51**

## 1 Introduction

In several industries such as oil, gas, electricity, railways, hospitals and airlines, firms have been regulated by price-caps. Especially, we can observe price-cap regulations in the telecommunications industries of many countries and regions.[1] In this paper, we discuss the effect of a change in a price-cap level on market outcomes. In our model, firms engage in Cournot or Stackelberg competition and the market price of the good must be less than a certain price-cap level.

A large number of theoretical works discuss the price-cap regulation in the contexts of monopoly and oligopoly.[2] Earle et al. (2007) consider Cournot oligopoly models where the demand is deterministic or stochastic and the price

[1] See, for instance, Sappington (2002).

[2] See, for example, Laffont and Tirole (1993) and Armstrong and Sappington (2007) with regard to examination of the effect of the price-cap in a monopoly market.

cap level is more than or equal to the competitive price. They show that social welfare is nonincreasing in the price cap level as long as the level is more than the competitive price in the deterministic case.[3] The model of Earle et al. (2007) is based on that of Roberts and Sonnenshein (1976); that is, they do not require the strict concavity of the profit function of a firm, which is assumed in Cournot models of almost previous works.[4] Instead of relaxing the assumption, they restrict their attention to symmetric equilibria. However, as is shown later in this paper, there can be asymmetric Cournot equilibria under a price-cap even if we assume the strict concavity of the profit functions.[5] Moreover, when we focus on asymmetric equilibria, a decrease in the price-cap level can be socially harmful even if the price cap level is more than the competitive price and there is no uncertainty.

We discuss simple Cournot and Stackelberg duopoly models with symmetric firms and a strictly convex cost function. First, we characterize the symmetric and asymmetric Cournot equilibria with a price-cap. We focus on the price-cap levels that are not below the competitive price. If the price-cap level is below the Cournot equilibrium price without any price-cap, then the price-cap level is binding; that is, the market price is equal to the price-cap level in any equilibrium. Moreover, there is a unique symmetric equilibrium. However, there are also asymmetric equilibria; that is, the output of a firm is larger than that of the other firm in some equilibria. We characterize each equilibrium as a convex combination of the symmetric equilibrium and the most asymmetric equilibrium where a coefficient of the convex combination represents the degree of symmetry of the equilibrium. By using the characterization, we provide comparative statics analyses. Unlike the comparative statics results of a monopoly model, we show that tightening or imposing a price-cap may decrease the equilibrium output of a firm and increase the profit of a firm.

Second, we consider a Stackelberg game with a price-cap. We show that the Stackelberg equilibrium is unique for any price-cap levels. Moreover, if the price-cap level is binding, then the Stackelberg equilibrium outputs are equal to the most asymmetric Cournot equilibrium. We give several comparative statics results of the Stackelberg equilibrium. The results are also different from those of a monopoly model.

Third, we consider an endogenous timing duopoly model à la Hamilton and Slutsky (1990) where Cournot competition or Stackelberg competition is

[3] Earle et al. (2007) show that if demand is uncertain, a reduction of the price cap level can decrease social welfare. See also Grimm and Zőttl (2010) on related results. Roques and Savva (2009) consider the effect of price caps on investment in Cournot oligopoly with uncertainty. Reynolds and Rietzke (2012) discuss an oligopoly model with endogenous entry. Corchón and Marcos (2012) consider the model where a regulater faces uncertainty about the marginal costs of firms.

There also exist several previous works that consider a Bertrand model with a price cap regulation. See, for example Bhaskar (1997) and Matsumura and Matsushima (2003).

[4] See also Milgrom and Roberts (1994) and Acemoglu and Jensen (2013) on related imperfect competition models.

[5] Earle et al. (2007) assume that the marginal cost of a firm is constant. On the other hand, we consider a convex cost function. Therefore, the model of Earle et al. (2007) is not a generalization of ours and vice versa.

achieved depending on the choices of the firms in a preplay stage. We show that the Cournot competition is achieved in the combination of the weakly dominant strategies of the firms in the preplay stage. Amir and Grilo (1999) considers a similar model but without any regulation and shows that the Cournot competition is realized in the combination of the weakly dominant strategies. Thus, our result shows the robustness of that of Amir and Grilo (1999).

Finally, we consider the welfare effect of the price-cap. First, since the cost functions are assumed to be identical and strictly convex, for any given price-cap level social welfare is increasing in the degree of symmetry of the equilibrium. Thus, if the price-cap level is binding in the Stackelberg competition, then social welfare under the Stackelberg competition is less than or equal to that under any Cournot equilibrium. Second, we show that if the price-cap level is equal to the competitive price, then the first best outcome is achieved in both market structures. Third, when we focus only on the symmetric Cournot equilibrium, social welfare is increased by a decrease in a price-cap level. Fourth, if the price-cap level is sufficiently close to the competitive price, then social welfare is also increased by a decrease in a price-cap level. However, if the price-cap level is sufficiently far from the competitive price, then social welfare can be decreased by a reduction in a binding price-cap level. This result is satisfied when we focus on the Stackelberg equilibrium and a sufficiently asymmetric Cournot equilibrium.

## 2 Model and Competitive Equilibrium

We assume that there are exactly two firms 1 and 2 that decide their output denoted by $x_i$ for $i = 1, 2$. Let $X = x_1 + x_2$. The cost functions of the firms are identical and given by $C(x_i)$ that satisfies $C' > 0$ and $C'' > 0$; that is, $C(x_i)$ is strictly convex. The inverse demand function of this market is given by $P(X)$ satisfying $P''X + P' < 0$ for all $X > 0$. Thus, the profit of firm $i = 1, 2$ is $P(X)x_i - C(x_i)$. Note that by assumptions above, the profit function of $i$ is strictly concave with $x_i$.

To characterize the Cournot and Stackelberg equilibria, we firstly derive the competitive equilibrium without any price caps. In this case, each firm decides to maximize their profit given the market price. Therefore, the first order condition of $i$ is $P(X) - C'(x_i) = 0$. Let $\varnothing^C(X) = C'^{-1}(P(X))$. Moreover, let $r^C(x_j)$ be the solution of $x_i = \varnothing^C(x_i + x_j)$. The competitive equilibrium output of a firm is $x^C = \varnothing^C(X^C)$ where $X^C = 2\varnothing^C(X^C)$. Since $\varnothing^C(X)$ is smooth and $\varnothing^{C\prime} \leq 0$, the equilibrium uniquely exists. We call $P(X^C)$ the competitive price.

## 3 Cournot Equilibria

Without loss of generality, in this section, we assume that $x_1 \geq x_2$ in any equilibrium. At first, we derive the Cournot equilibria without any price caps.

The first order condition of $i$ is

$$P'(X)x_i + P(X) - C'(x_i) = 0.$$

This condition yields the best reply function of firm $i$ be $r(x_j)$ for $i \neq j$. The cumulative best reply function $\varnothing(X)$ that is the optimal output of a firm which is consistent with an aggregate output $X$. Formally, $\varnothing(X)$ is the unique solution of $x_i = r(X - x_i)$. The Cournot equilibrium output of a firm is $x^N = \varnothing(X^N)$ where $X^N = 2\varnothing(X^N)$. Since $\varnothing$ is smooth and $\varnothing' \leq 0$, the equilibrium uniquely exists.[6] See Vives (2001, Ch.4) on the complete proof of this result. By comparing the competitive case, we have $\varnothing(X) < \varnothing^C(X)$ for all $X$ satisfying $\varnothing^C(X) > 0$. Therefore, we have $X^N < X^C$ and $\varnothing(X) < \varnothing^C(X)$ for all $X \leq X^C$.

Next, we consider Cournot equilibria with a price-cap $\bar{P}$. Thus, the profit of the firm is given by

$$\min\left\{P(X), \bar{P}\right\} x_i - C(x_i).$$

Let $\bar{X}$ be such that $P(\bar{X}) = \bar{P}$. Throughout this paper, we focus on the price cap level over the competitive equilibrium price: $\bar{P} \geq P(X^C)$.

First, suppose $\bar{P} \geq P(X^N)$. Then, $\left(x^N, x^N\right)$ is the unique equilibrium. That is, if the price-cap level is more than or equal to the Cournot equilibrium price without any price-cap, then the price-cap level is not binding and thus imposing the price-cap does not affect the equilibrium as long as $\bar{P} \geq P(X^N)$.

Second, suppose $\bar{P} \in \left(P(X^N), P(X^C)\right]$. At first, consider $(x_1, x_2)$ that satisfies $\bar{P} > P(x_1 + x_2)$. Since $\left(x^N, x^N\right)$ is the unique Cournot equilibrium without any price caps, either of the firms has the incentive to change its output. Therefore, a pair $(x_1, x_2)$ that satisfies $\bar{P} > P(x_1 + x_2)$ is not an equilibrium and thus $\bar{P}$ is binding in any equilibrium; that is, $\bar{P} = P(X^N\left(\bar{P}\right))$ and $X^N\left(\bar{P}\right) = \bar{X}$.

Now, consider $(x_1, x_2)$ that satisfies $x_1 + x_2 = \bar{X}$. We derive the condition that $(x_1, x_2)$ is a pair of equilibrium outputs. First, firm $i$ has no incentive to increase its output if and only if

$$P'(\bar{X})x_i + P(\bar{X}) - C'(x_i) \leq 0 \Rightarrow x_i \geq \varnothing(\bar{X}). \tag{1}$$

On the other hand, firm $i$ has no incentive to decrease its output if and only if

$$P(\bar{X}) - C'(x_i) \geq 0 \Rightarrow x_i \leq \varnothing^C(\bar{X}). \tag{2}$$

This is because, any decrease of firm $i$'s output never raises the price. Therefore, the equilibrium output of firm $i$ must satisfy $\varnothing(\bar{X}) \leq x_i \leq \varnothing^C(\bar{X})$.

To completely characterize the equilibrium, consider the symmetric equilibrium $\left(\bar{X}/2, \bar{X}/2\right)$ and the most asymmetric equilibrium $\left(\overline{x}^N\left(\bar{P}\right), \underline{x}^N\left(\bar{P}\right)\right)$ where

$$\begin{aligned}
\overline{x}^N\left(\bar{P}\right) &= \min\{\bar{X} - \varnothing(\bar{X}), \varnothing^C(\bar{X})\}, \\
\underline{x}^N\left(\bar{P}\right) &= \max\{\varnothing(\bar{X}), \bar{X} - \varnothing^C(\bar{X})\}.
\end{aligned}$$

[6] Earle et al. (2007) discuss a model where multiple symmetric equilibria can exist and focus only on the symmetric equilibria. On the other hand, although we do not consider the case of multiple symmetric equilibria, we focus on both the symmetric and asymmetric equilibria.

Then, as will be shown later, the equilibrium output is represented by the convex combination of the two equilibria. Thus, let

$$\begin{aligned} x_B^N\left(\bar{P},\alpha\right) &= \alpha\bar{X}/2+(1-\alpha)\,\overline{x}^N\left(\bar{P}\right), \\ x_S^N\left(\bar{P},\alpha\right) &= \alpha\bar{X}/2+(1-\alpha)\,\underline{x}^N\left(\bar{P}\right) \end{aligned}$$

for $\bar{P} \in \left[P(X^N), P(X^C)\right]$ and $\alpha \in [0,1]$, and $x_B^N\left(\bar{P},\alpha\right) = x_S^N\left(\bar{P},\alpha\right) = x^N$ for $\bar{P} > P(X^N)$ and $\alpha \in [0,1]$. Note that $x_B^N\left(\bar{P},\alpha\right) + x_S^N\left(\bar{P},\alpha\right) = \bar{X}$ and $x_B^N\left(\bar{P},\alpha\right) \geq x_S^N\left(\bar{P},\alpha\right)$ for any $\bar{P} \in \left[P(X^N), P(X^C)\right]$ and $\alpha \in [0,1]$. If $\alpha = 1$, then the equilibrium is symmetric. Moreover, if $\alpha = 0$, then the equilibrium is the most asymmetric one. Therefore, $\alpha$ represents the degree of symmetry of the equilibrium. Note that $\partial x_B^N\left(\bar{P},\alpha\right)/\partial\alpha < 0$ and $\partial x_S^N\left(\bar{P},\alpha\right)/\partial\alpha > 0$ for $\bar{P} \in \left(P(X^C), P(X^N)\right)$.

In sum, we have the following result.

**Theorem 1** *A pair of the outputs of the firms is an equilibrium if and only if* $(x_1,x_2) = \left(x_B^N\left(\bar{P},\alpha\right), x_S^N\left(\bar{P},\alpha\right)\right)$ *for* $\alpha \in [0,1]$. *Moreover,* $\underline{x}^N\left(\bar{P}\right)$ *is a U-shaped function and* $\overline{x}^N\left(\bar{P}\right)$ *is an inverted U-shaped function of* $\bar{P} \in \left(P(X^N), P(X^C)\right)$. *In addition, the minimizer of* $\underline{x}^N\left(\bar{P}\right)$, *denoted* $\bar{P}^*$, *is equivalent to the maximizer of* $\overline{x}^N\left(\bar{P}\right)$ *and* $\bar{P}^* \in \left(P(X^N), P(X^C)\right)$.

The proof of this is provided in the Appendix.

By this result, if $\bar{P} \in \left(P(X^N), P(X^C)\right)$, then $x^N\left(\bar{P}\right)$ is not unique and there are asymmetric equilibria. This is because if $x_1 + x_2 > X^N$ and there is no price-cap, then firm 2 has an incentive to reduce its output. However, if $x_1 + x_2 \in \left(X^N, \bar{X}\right]$, then firm 2 may not have the incentive because any reduction does not raise the price. Therefore, there are multiple equilibria under $\bar{P} \in \left(P(X^N), P(X^C)\right)$.

Next, we consider the effect of changes in $\bar{P}$ and $\alpha$ on the market outcomes. It should be noted that $x_B^N\left(\bar{P},\alpha\right)$ and $x_S^N\left(\bar{P},\alpha\right)$ is not differentiable at $\bar{P}^*$ for all $\alpha < 1$. However, since they are differentiable at the other points, we will use differential representations such as $\partial x_B^N\left(\bar{P},\alpha\right)/\partial\bar{P}$ and $\partial x_S^N\left(\bar{P},\alpha\right)/\partial\alpha$.

At first, we focus on the most asymmetric equilibrium $(x_1,x_2) = \left(\overline{x}^N\left(\bar{P}\right), \underline{x}^N\left(\bar{P}\right)\right) = \left(x_B^N\left(\bar{P},\alpha\right), x_S^N\left(\bar{P},\alpha\right)\right)$. By Theorem 1, $\overline{x}^N\left(\bar{P}\right)$ and $\underline{x}^N\left(\bar{P}\right)$ are a U-shaped and an inverted U-shaped functions of $\bar{P}$, respectively. First, suppose $\bar{P} \in \left[P(X^C), \bar{P}^*\right)$; that is, $\bar{X}$ is relatively large. If the total production is equal to $\bar{X}$, then no firm changes the production as far as $\varnothing(\bar{X}) \leq x_i \leq \varnothing^C(\bar{X})$. Note that both $\varnothing(\bar{X})$ and $\varnothing^C(\bar{X})$ are decreasing in $\bar{X}$ (increasing in $\bar{P}$). In this case, $\varnothing^C(\bar{X}) < \bar{X} - \varnothing(\bar{X})$ because $\bar{X}$ is large. Thus, in the most asymmetric equilibrium, firm 1 produces $\varnothing^C(\bar{X})$ and firm 2 produces the remainder: $\bar{X} - \varnothing^C(\bar{X})$. Second, suppose $\bar{P} \in \left[\bar{P}^*, P(X^N)\right]$; that is, $\bar{X}$ is relatively small. In this case, $\varnothing(\bar{X}) > \bar{X} - \varnothing^C(\bar{X})$ because $\bar{X}$ is small. Thus, if $(x_1,x_2) = \left(\varnothing^C(\bar{X}), \bar{X} - \varnothing^C(\bar{X})\right)$, then firm 2 has an incentive to increase its output. Thus, in the most asymmetric equilibrium, firm 2 produces $\varnothing(\bar{X})$ and firm 1 produces the remainder: $\bar{X} - \varnothing(\bar{X})$. Since both $\varnothing(\bar{X})$ and $\varnothing^C(\bar{X})$ are decreasing in $\bar{X}$ (increasing in $\bar{P}$), both $\bar{X}- \varnothing^C(\bar{X})$ and $\bar{X}- \varnothing(\bar{X})$ are increasing in $\bar{X}$ (decreasing in $\bar{P}$). Therefore, $\overline{x}^N\left(\bar{P}\right)$ and $\underline{x}^N\left(\bar{P}\right)$ are a U-shaped and an inverted U-shaped functions of $\bar{P}$, respectively.

By using the characterization of the Cournot equilibria, we provide comparative statics results. Since there are multiple equilibria, we fix $\alpha$ in this analysis. Note that, if we focus on the symmetric equilibrium, $\alpha$ is fixed to be 1. Then, we immediately have the following results on the equilibrium output.

**Corollary 1** *1. If $\bar{P} \in \left[P(X^C), \bar{P}^*\right)$ and $\alpha$ is sufficiently small, then $\partial x_B^N\left(\bar{P},\alpha\right)/\partial\bar{P} > 0$.*

*2. If $\bar{P} \in \left(\bar{P}^*, P(X^N)\right)$ and $\alpha$ is sufficiently small, then $\partial x_S^N\left(\bar{P},\alpha\right)/\partial\bar{P} > 0$.*

The first and second results are obvious from the facts that $\underline{x}^N\left(\bar{P}\right)$ is a U-shaped function and $\overline{x}^N\left(\bar{P}\right)$ is an inverted U-shaped function. If $\bar{P} \in \left[P(X^C), P(X^N)\right)$, then the total output is equal to $\bar{X}$ and thus decreasing in $\bar{P}$. However, if $\alpha$ is sufficiently small, then a rise in $\bar{P}$ increases the equilibrium output of either firm.

The first and second results of Corollary 1 imply that a reduction of $\bar{P}$ may decrease the output of a firm. To provide an intuition of this result, suppose $\alpha = 0$. First, we focus on $x_B^N\left(\bar{P},0\right)$; the equilibrium output of a firm that is larger than that of the other firm. As is shown earlier, $x_B^N\left(\bar{P},0\right)$ is increasing in $\bar{P}$ if and only if $\bar{P} \in \left[P(X^C), \bar{P}^*\right)$; that is, $x_B^N\left(\bar{P},0\right) = \varnothing^C(\bar{X})$. This function represents the optimal output level of a price taker where the price is given by $\bar{P}$ and thus it is increasing in $\bar{P}$ (decreasing in $\bar{X}$ ). Thus, $x_B^N\left(\bar{P},0\right)$ is increasing in $\bar{P}$ if $\bar{P} \in \left[P(X^C), \bar{P}^*\right)$. Second, we focus on $x_S^N\left(\bar{P},0\right)$; the equilibrium output of a firm that is smaller than that of the other firm. As is shown earlier, $x_S^N\left(\bar{P},0\right)$ is increasing in $\bar{P}$ if and only if $\bar{P} \in \left(\bar{P}^*, P(X^N)\right)$; that is, $x_S^N\left(\bar{P},0\right) = \varnothing^N(\bar{X})$. This is the best reply function where the price is given by $\bar{P}$ and thus it is increasing in $\bar{P}$ (decreasing in $\bar{X}$). These comparative statics results are sharply contrast to that of a usual monopoly model. In a monopoly model, a decrease in a binding price-cap level increases the output of a monopolist, because the market price is not risen by a reduction of the output. However, in our duopoly model, this result may not be satisfied. Moreover, since $\partial x_B^N\left(\bar{P},1\right)/\partial\bar{P} = \partial x_S^N\left(\bar{P},1\right)/\partial\bar{P} \leq 0$, the results are dependent on the degree of symmetry $\alpha$.

**Example 1**

Let $P(X) = 1 - X/2$ and $C(x_i) = x_i^2/2$. Then, $X^N = 4/5$ and $P(X^N) = 3/5$, and $X^C = 1$ and $P(X^C) = 1/2$, and $\bar{P}^* = 6/11$. In addition,

$$\begin{aligned} \varnothing\left(\bar{X}\right) &= 2/3 - \bar{X}/3, \varnothing^C(\bar{X}) = 1 - \bar{X}/2, \\ \bar{X} - \varnothing(\bar{X}) &= 4\bar{X}/3 - 2/3, \bar{X} - \varnothing^C(\bar{X}) = 3\bar{X}/2 - 1. \end{aligned}$$

Figure 1 depicts $\overline{x}^N\left(\bar{P}\right)$ and $\underline{x}^N\left(\bar{P}\right)$ of this example.As illustrated in the Figure 1, the range of the equilibrium outputs; $\overline{x}^N\left(\bar{P}\right) - \underline{x}^N\left(\bar{P}\right)$, is increasing between $P(X^C)$ and $\bar{P}^*$ and is decreasing between $\bar{P}^*$ and $P(X^N)$.

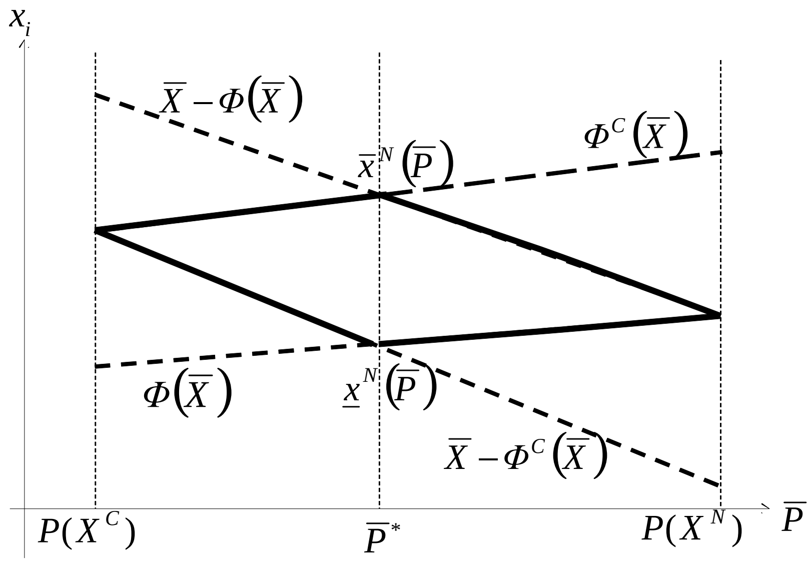


Figure 1: $\overline{x}^N\left(\bar{P}\right)$,$\underline{x}^N\left(\bar{P}\right)$

**Example 2**

Let $P(X) = 1 - X^{10}$ and $C(x_i) = x_i^2/2$. Then, $X^N \approx 0.795$ and $P(X^N) \approx 0.900$, and $X^C \approx 0.939$ and $P(X^C) \approx 0.469$. Moreover,

$$\begin{aligned} \varnothing(\bar{X}) &= \frac{1-\bar{X}^{10}}{10\bar{X}^9+1}, \varnothing^C(\bar{X}) = 1-\bar{X}^{10}, \\ \bar{X} - \varnothing(\bar{X}) &= \bar{X} - \frac{1-\bar{X}^{10}}{10\bar{X}^9+1}, \bar{X} - \varnothing^C(\bar{X}) = \bar{X} - \left(1-\bar{X}^{10}\right). \end{aligned}$$

Thus, $\bar{P}^* \approx 0.714$ and the corresponding total output is about 0.882. We can write the similar graphs of $\overline{x}^N\left(\bar{P}\right)$ and $\underline{x}^N\left(\bar{P}\right)$ to those of Figure 1.

Next, we focus on the profits of the firms. We have the following result. Let the Cournot equilibrium profit under $\left(\bar{P}, \alpha\right)$ be

$\pi_J^N\left(\bar{P}, \alpha\right) = P(x_B^N\left(\bar{P}, \alpha\right) + x_S^N\left(\bar{P}, \alpha\right))x_J^N\left(\bar{P}, \alpha\right) - C(x_J^N\left(\bar{P}, \alpha\right))$ for $J = B, S$.

**Corollary 2** *Suppose $\bar{P} \in \left(P(X^C), P(X^N)\right)$. Then,*

1. $\pi_B^N\left(\bar{P}, \alpha\right) > \pi_S^N\left(\bar{P}, \alpha\right)$ *for all* $\alpha \in [0, 1)$.
2. $\partial\pi_B^N\left(\bar{P}, \alpha\right)/\partial\alpha < 0$ *and* $\partial\pi_S^N\left(\bar{P}, \alpha\right)/\partial\alpha > 0$ *for all* $\alpha \in [0, 1)$.
3. *If* $\bar{P} \in \left(\bar{P}^*, P(X^N)\right)$ *and* $\alpha$ *is sufficiently small, then* $\partial\pi_B^N\left(\bar{P}, \alpha\right)/\partial\bar{P} < 0$ *can be satisfied.*

4. *If $\bar{P} \in \left[P(X^C), \bar{P}^*\right)$ and $\alpha$ is sufficiently small, then $\partial \pi_S^N \left(\bar{P}, \alpha\right) / \partial \bar{P} < 0$ can be satisfied.*

The proof of the first and second results are provided in the Appendix. Here, we show that $\partial \pi_J^N \left(\bar{P}, \alpha\right) / \partial \bar{P}$ can be negative for $J = B, S$. We reconsider Example 2. Suppose $\alpha = 0$. We have

$$\begin{aligned} \pi_B^N (0.88, 0) &\approx 0.297 > 0.289 \approx \pi_B^N (0.89, 0), \\ \pi_S^N (0.64, 0) &\approx 0.134 > 0.131 \approx \pi_S^N (0.65, 0). \end{aligned}$$

This is because the reduction of the output of a firm by an increase of $\bar{P}$ can be large when $\alpha$ is small. Thus, $\partial \pi_J^N \left(\bar{P}, \alpha\right) / \partial \bar{P}$ can be negative for $J = B, S$.

Note that, if $\alpha = 1$, then $\partial \pi_B^N \left(\bar{P}, \alpha\right) / \partial \bar{P} \geq 0$ and $\partial \pi_S^N \left(\bar{P}, \alpha\right) / \partial \bar{P} \geq 0$. Therefore, the comparative statics results are dependent on $\alpha$.

# 4 Stackelberg Equilibrium

We consider the case where two firms engage in Stackelberg competition where a firm is a leader and the other is a follower. Let $L$ and $F$ represent the leader and follower, respectively, where $L, F = 1, 2$ and $L \neq F$. First, we derive the Stackelberg equilibrium without any price caps. Firm $L$'s maximization problem is

$$\max_{x_L} P(x_L + r(x_L))x_L - C(x_L).$$

The first order condition of firm $L$ is

$$P(X) + P'(X)(1 + r'(x_L))x_L - C'(x_L) = 0. \tag{3}$$

Moreover, we assume that the second order condition is satisfied for all $x_L$. Let $x_L^S$ and $x_F^S$ be the Stackelberg equilibrium outputs of the leader and the follower without any price caps. Then, $\left(x_L^S, x_F^S\right)$ is unique, and $x_L^S$ satisfies (3) and $x_F^S = r\left(x_L^S\right)$. Moreover, let $X^S = x_L^S + x_F^S$. We give three well-known facts on the Stackelberg equilibrium. First, $x_L^S > x_F^S$. Second, the profit of the leader in the Stackelberg equilibrium is larger than that in the Cournot equilibrium. Third, $P\left(X^N\right) > P\left(X^S\right) > P\left(X^C\right)$ or equivalently $X^C > X^S > X^N$. See, for instance, Etro (2008) and Ino and Matsumura (2011) on the proofs of these results.

Next, we consider the Stackelberg equilibrium with a price-cap $\bar{P}$. We derive the best reply of the follower $r_F\left(x_L, \bar{P}\right)$ under $\bar{P}$. First, if $\bar{X} - x_L \leq r(x_L)$, then $r_F\left(x_L, \bar{P}\right) = r(x_L)$ because $\bar{P}$ is not binding for $(x_L, r(x_L))$. Second, if $\bar{X} - x_L \geq r(x_L)$, then $\bar{P}$ is binding for $(x_L, r(x_L))$. Therefore, if $\bar{X} - x_L \in \left[r(x_L), r^C(x_L)\right]$, then $r_F\left(x_L, \bar{P}\right) = \bar{X} - x_L$. This is because any decrease of the follower's output does not increase the market price. Recall (1) and (2) with regard to this point. Moreover, if $\bar{X} - x_L \geq r^C(x_L)$, $r_F\left(x_L, \bar{P}\right) = r^C(x_L)$. This is because, in this case, any decrease of the follower's output does not increase the market price. In sum, $r_F\left(x_L, \bar{P}\right)$ is given by

$$\begin{aligned}
r_F\left(x_L, \bar{P}\right) &= r\left(x_L\right) \text{ if } \bar{X} - x_L \leq r\left(x_L\right), \\
&= \bar{X} - x_L \text{ if } \bar{X} - x_L \in \left[r\left(x_L\right), r^C\left(x_L\right)\right], \\
&= r^C\left(x_L\right) \text{ if } \bar{X} - x_L \geq r^C\left(x_L\right).
\end{aligned}$$

Then, we have the following result.

**Theorem 2** *If $\bar{P} > P\left(X^S\right)$, then $\left(x_L^S\left(\bar{P}\right), x_F^S\left(\bar{P}\right)\right) = \left(x_L^S, x_F^S\right)$. If $\bar{P} \in \left[P\left(X^C\right), P\left(X^S\right)\right]$, then $\left(x_L^S\left(\bar{P}\right), x_F^S\left(\bar{P}\right)\right) = \left(x_B^N\left(\bar{P}, 0\right), x_S^N\left(\bar{P}, 0\right)\right)$.*

The proof of this is provided in the Appendix.

As shown in the previous section, there can be multiple Cournot equilibria but the Stackelberg equilibrium is always unique. Moreover, if $\bar{P} \leq P\left(X^S\right)$ $(< P\left(X^N\right))$, then the Stackelberg equilibrium is equal to the most asymmetric Cournot equilibrium. If $\bar{P}$ is binding, then the leader has an incentive to increase its production as long as the output is less than $\varnothing^C(\bar{X})$. However, if the production of the leader is more than $\bar{X} - \varnothing(\bar{X})$, then the follower has an incentive to choose its production more than $\varnothing(\bar{X})$. Since the leader wants to keep the price-cap binding, it chooses $x_L^S\left(\bar{P}\right) = x_B^N\left(\bar{P}, 0\right) = \min\{\varnothing^C(\bar{X}), \bar{X} - \varnothing(\bar{X})\}$ and thus the follower chooses $x_F^S\left(\bar{P}\right) = x_S^N\left(\bar{P}, 0\right) = \min\{\varnothing(\bar{X}), \bar{X} - \varnothing^C(\bar{X})\}$.

Next, we give comparative statics results on the Stackelberg equilibrium. In Corollaries 1 and 2, we provide some counterintuitive results when we focus on a sufficiently asymmetric Cournot equilibrium. By Theorem 2, we have the results as a comparative statics analysis of the unique equilibrium.

**Corollary 3** *1. $\bar{P}^* < P(X^S)$.*

*2. If $\bar{P} \in \left[P(X^C), \bar{P}^*\right)$, then $x_L^{S\prime}\left(\bar{P}\right) > 0$. If $\bar{P} \in \left[\bar{P}^*, P(X^S)\right)$, then $x_L^{S\prime}\left(\bar{P}\right) < 0$.*

*3. If $\bar{P} \in \left[\bar{P}^*, P(X^S)\right)$, then $x_F^{S\prime}\left(\bar{P}\right) > 0$. If $\bar{P} \in \left[P(X^C), \bar{P}^*\right)$, then $x_F^{S\prime}\left(\bar{P}\right) < 0$.*

The first result of Corollary 3 implies $P(X^C) < \bar{P}^* < P(X^S) < P\left(X^N\right)$. Since $x_L^S\left(\bar{P}\right) = x_B^N\left(\bar{P}, 0\right)$ and $x_F^S\left(\bar{P}\right) = x_S^N\left(\bar{P}, 0\right)$, we have the second and third results. This implies that if $\bar{P}$ is binding, then a decrease in $\bar{P}$ increases the output of a firm but decrease that of the other.

Next, we consider the profits of the firms.

**Corollary 4** *1. If $\bar{P} \in \left(\bar{P}^*, P(X^S)\right)$, then $\pi_L'\left(\bar{P}\right) < 0$ can be satisfied.*

*2. If $\bar{P} \in \left[P(X^C), \bar{P}^*\right)$, then $\pi_F'\left(\bar{P}\right) < 0$ can be satisfied.*

These results are obvious from Corollary 2, $\pi_L\left(\bar{P}\right) = \pi_B\left(\bar{P}, 0\right)$ and $\pi_F\left(\bar{P}\right) = \pi_S\left(\bar{P}, 0\right)$. Therefore, $\pi_I'\left(\bar{P}\right)$ can be negative for $I = L, F$.

# 5 Endogenous Timing

We consider an endogenous timing duopoly model à la Hamilton and Slutsky (1990) called an observable delay game.[7] In the first stage, the firms simultaneously chooses $l$ or $f$. On the one hand, if the firms choose the same strategy in the first stage, then they engage in Cournot competition in the second stage. On the other hand, if the firms choose different strategies, they engage in Stackelberg competition where the firm that chooses $l$ is a leader and the other firm is a follower. Thus, the games in the second stage have already been discussed in the previous sections. We focus on the subgame perfect equilibria. We have the following result.

**Theorem 3** *For any $\bar{P} \geq P\left(X^C\right)$, $l$ is the weakly dominant strategy of each firm.*

The proof of this is provided in the Appendix.

Amir and Grilo (1999, Theorem 2.2 and Corollary 2.5) show that when there is no price-cap, $l$ is the dominant strategy of each firm. If $\bar{P} \geq P\left(X^N\right)$; that is, if $\bar{P}$ is not binding in any cases, then we can apply their result. Next, if $\bar{P} \in \left[P(X^C), P(X^S)\right)$; that is, if $\bar{P}$ is binding in both cases, then $\pi_F\left(\bar{P}\right) = \pi_S\left(\bar{P}, 0\right) \leq \pi_S\left(\bar{P}, \alpha\right)$, because of Corollary 2 and Theorem 2. Finally, suppose $\bar{P} \in \left[P(X^S), P\left(X^N\right)\right)$; that is, if $\bar{P}$ is binding only in the case of Cournot compeition. Since $\bar{P}^* < P(X^S)$ and Corollary 1 holds, $\partial x_S^N\left(\bar{P}, 0\right)/\partial\bar{P} > 0$. This implies $\pi_F\left(\bar{P}\right) \leq \pi_S\left(\bar{P}, \alpha\right)$. Therefore, we have Theorem 1.

By this result, the Cournot competition is realized in the second stage as a pair of the weakly dominant strategies. Our result implies that the result of Amir and Grilo (1999) continues to hold even if there is a price-cap regulation whose level is not less than $P(X^C)$.

# 6 Welfare Effects

We consider the welfare effect of a change in $\bar{P}$. Social welfare is defined as the simple sum of the consumers' and producers' surplus; i.e.,

$$\begin{aligned} W(x_1, x_2) &= \int_0^X P(Z)dZ - P(X)X + \pi_1 + \pi_2 \\ &= \int_0^X P(Z)dZ - C(x_1) - C(x_2). \end{aligned}$$

First, we examine the effect of a change in $\alpha$ on social welfare for given $\bar{P} \in \left(P(X^C), P(X^N)\right)$. Then, we have the following result.

[7] See also Matsumura (1995, 2003), Pal (1998), Amir and Grilo (1999), van Damme and Hurkens (1999) and Matsumura and Ogawa (2009) on the applications of the observable delay game.

**Proposition 1** *For any given* $\bar{P} \in \left(P(X^C), P(X^N)\right)$, $dW\left(x_B^N\left(\bar{P},\alpha\right), x_S^N\left(\bar{P},\alpha\right)\right)/d\alpha < 0$ *for all* $\alpha \in [0,1]$. *In addition, for any given* $\bar{P} \in \left(P(X^C), P(X^S)\right)$, $W\left(x_L^S\left(\bar{P}\right), x_F^S\left(\bar{P}\right)\right) < W\left(x_B^N\left(\bar{P},\alpha\right), x_S^N\left(\bar{P},\alpha\right)\right)$ *for all* $\alpha \in (0,1]$ *and they are equivalent for* $\alpha = 0$.

Since we assume a strictly convex cost function; that is, $C''(\cdot) > 0$, the symmetric equilibrium is the most efficient on for any $\bar{P} \in \left(P(X^C), P(X^N)\right)$. Moreover, a decrease of the degree of symmetry increases $x_B^N\left(\bar{P},\alpha\right) - x_S^N\left(\bar{P},\alpha\right)$ and the total cost of this market. Therefore, it reduces social welfare.

Since the Stackelberg equilibrium is equal to the most asymmetric Cournot equilibrium, social welfare of the Stackelberg equilibrium is less than or equal to that of a Cournot equilibrium as long as $\bar{P}$ is binding. This implies an efficient outcome is achieved as a result of the endogenous timing game discussed in the previous section as long as $\bar{P} \in \left(P(X^C), P(X^S)\right)$.

Second, we have the following result on the first best outcome.

**Proposition 2** *If* $\bar{P} = P(X^C)$, *then the first best outcome is achieved in both Cournot and Stackelberg equilibria; that is,*

$$W(x_B^N\left(\bar{P},\alpha\right), x_S^N\left(\bar{P},\alpha\right)) = W(x_L^S\left(\bar{P}\right), x_F^S\left(\bar{P}\right)) = W(x^C, x^C) \geq W(x_1, x_2)$$

*for all* $x_1, x_2$ *and* $\alpha \in [0,1]$.

By Corollary 1, this proposition is straightforward. Note that the output floor also yields the first-best outcome if the output floor level is equal to $X^C$. See Matsumura and Okumura (2013) on this fact.

Proposition 2 implies that the authority should set the price-cap level equal to the competitive price. However, in the real world, it may be difficult to know the exact competitive price. Therefore, we also focus on the price-cap level between the competitive price and the Cournot equilibrium price without any price-caps.

We focus on social welfare under $x_B^N\left(\bar{P},\alpha\right), x_S^N\left(\bar{P},\alpha\right)$ for given $\alpha \in [0,1]$. Moreover, since any changes in $\bar{P}$ do not change the market outcomes as long as $\bar{P} > P(X^N)$, we focus on $\bar{P} \in \left[P(X^C), P(X^N)\right]$. Note that, this analysis includes the case of the Stackelberg competition with a binding $\bar{P}$ as a special case.

First, we restrict our attention to the case where the degree of symmetry is sufficiently large.

**Proposition 3** *Suppose* $\alpha$ *is sufficiently large. If* $\bar{P} \in \left[P(X^C), P(X^N)\right)$, *then* $dW\left(x_B^N\left(\bar{P},\alpha\right), x_S^N\left(\bar{P},\alpha\right)\right)/d\bar{P} < 0$.

It is sufficient to consider the case where $\alpha = 1$. Fix $\bar{P} \in \left[P(X^C), P(X^N)\right)$. Then, $\left(x_B^N\left(\bar{P},1\right), x_S^N\left(\bar{P},1\right)\right) = (\bar{X}/2, \bar{X}/2)$ and social welfare is

$$W(\bar{X}/2, \bar{X}/2) = \int_0^{\bar{X}} P(Z)dZ - 2C(\bar{X}/2).$$

Therefore, $dW(\bar{X}/2, \bar{X}/2)/d\bar{X} = P(\bar{X}) - C'(\bar{X}) > 0$ as long as $\bar{X} < X^C$.

Note that Earle et al. (2007) assume that each firm has a constant marginal cost and focus only on the symmetric equilibria. They show $dW\left(x_B^N\left(\bar{P}, \alpha\right), x_S^N\left(\bar{P}, \alpha\right)\right)/d\bar{P} \leq 0$ for all $\bar{P} \geq P(X^C)$. On the other hand, we show that if an equilibrium is sufficiently symmetric and the cost function is convex, then $dW\left(x_B^N\left(\bar{P}, \alpha\right), x_S^N\left(\bar{P}, \alpha\right)\right)/d\bar{P} \leq 0$ for all $\bar{P} \geq P(X^C)$. Therefore, the result of Earle et al. (2007) is robust as long as we focus on a symmetric equilibrium.

This result implies that if the degree of symmetry is sufficiently large, any decrease in $\bar{P}$ is socially desirable as long as $\bar{P} > P(X^C)$, because a decrease in $\bar{P}$ increases the total production. However, if we focus on asymmetric equilibria, then some decrease in $\bar{P}$ may not be socially desirable.

Next, consider the case where the degree the degree of symmetry is small. First, we show that a decrease in $\bar{P}$ can reduce social welfare only if $\bar{P} \in \left[\bar{P}^*, P(X^N)\right)$.

**Proposition 4** *If $\bar{P} \in \left[P(X^C), \bar{P}^*\right)$, then $dW\left(x_B^N\left(\bar{P}, \alpha\right), x_S^N\left(\bar{P}, \alpha\right)\right)/d\bar{P} < 0$ for any $\alpha \in [0,1]$ and thus $dW\left(x_L^S\left(\bar{P}\right), x_F^S\left(\bar{P}\right)\right)/d\bar{P} < 0$.*

Proposition 4 implies that if the price-cap level has already been sufficiently near the competitive price, then making the price-cap level closer to the competitive level is socially desirable. In this case, a decrease in $\bar{P}$ reduces the difference between $x_B^N\left(\bar{P}, \alpha\right)$ and $x_S^N\left(\bar{P}, \alpha\right)$ for all $\alpha \in [0,1]$. Since we consider a strictly convex cost function, a reduction in the difference decreases the industry cost. Thus, in this case, decreasing the price-cap level is socially desirable.

By Propositions 2 and 3, if $\bar{P} \in \left[P(X^C), \bar{P}^*\right)$ or $\alpha$ is sufficiently large, then any decrease of $\bar{P}$ increase social welfare. However, in the other case, we have the following result.

**Proposition 5** *If $\bar{P} \in \left(\bar{P}^*, P(X^N)\right]$ and $\alpha$ is sufficiently small, then $dW\left(x_B^N\left(\bar{P}, \alpha\right), x_S^N\left(\bar{P}, \alpha\right)\right)/d\bar{P} > 0$ can be satisfied. Moreover, if $\bar{P} \in \left(\bar{P}^*, P(X^N)\right]$, then $dW\left(x_L^S\left(\bar{P}\right), x_F^S\left(\bar{P}\right)\right)/d\bar{P} > 0$ can be satisfied.*

If $\bar{P} \in \left(\bar{P}^*, P(X^N)\right]$ and $\alpha$ is sufficiently small, then a reduction of $\bar{P}$ increases the equilibrium output of a large firm and decreases that of a small firm. That is, in this case, a decrease in $\bar{P}$ expands the difference between $x_B^N\left(\bar{P}, \alpha\right)$ and $x_S^N\left(\bar{P}, \alpha\right)$. This implies, although a reduction of $\bar{P}$ increases the total output, it vitally increases the industry cost. This is because we consider a strictly convex cost function. Formally, we show this fact by reexamining the examples above.

First, we reconsider Example 1. In this example, the Stackelberg equilibrium without any price-cap is $\left(x_L^S, x_F^S\right) = (3/7, 11/28)$ and $P\left(X^S\right) = 33/56$. Next, if $\bar{P} \in \left(\bar{P}^*, P(X^N)\right] = (6/11, 3/5]$,

$$\frac{dW\left(x_B^N\left(\bar{P}, \alpha\right), x_S^N\left(\bar{P}, \alpha\right)\right)}{d\bar{P}} = \frac{100}{9}\bar{P}\alpha - \frac{20}{3}\alpha - \frac{86}{9}\bar{P} + \frac{10}{3}\alpha^2 - \frac{50}{9}\bar{P}\alpha^2 + \frac{16}{3}.$$

If $\alpha = 0$ and $\bar{P} \in (6/11, 24/43)$, then

$$\frac{dW\left(x_B^N\left(\bar{P}, 0\right), x_S^N\left(\bar{P}, 0\right)\right)}{d\bar{P}} = \frac{dW(x_L^S\left(\bar{P}\right), x_F^S\left(\bar{P}\right))}{d\bar{P}} > 0.$$

This result implies that if the firms engage in the Stackelberg competition, then a reduction of $\bar{P}$ can reduce social welfare. Moreover, if the firms engage in the Cournot competition and the degree of symmetry is sufficiently small, then a reduction of $\bar{P}$ can also reduce social welfare. However, in this example, any reduction of $\bar{P}$ increase social welfare if $\alpha$ is not very small; e.g., if $\alpha = 1/2$. In the next example, we show that even if $\alpha = 1/2$, then a reduction of $\bar{P}$ can also reduce social welfare.

Second, we reconsider Example 2. The equilibrium outcomes for several pairs of $(\alpha, \bar{P})$ are summarized as Table 1.

| $\alpha \setminus \bar{P}$ | 0.9 | 0.85 | 0.8 | 0.75 | 0.72 | 0.7 |
|---|---|---|---|---|---|---|
| 1 | $0.397, 0.397$ | $0.414, 0.414$ | $0.426, 0.426$ | $0.435, 0.435$ | $0.440, 0.440$ | $0.443, 0.443$ |
| 3/4 | $0.397, 0.397$ | $0.442, 0.386$ | $0.473, 0.379$ | $0.496, 0.375$ | $0.507, 0.373$ | $0.507, 0.379$ |
| 1/2 | $0.397, 0.397$ | $0.470, 0.385$ | $0.519, 0.333$ | $0.556, 0.315$ | $0.574, 0.306$ | $0.572, 0.315$ |
| 1/4 | $0.397, 0.397$ | $0.497, 0.330$ | $0.566, 0.286$ | $0.617, 0.254$ | $0.641, 0.239$ | $0.636, 0.251$ |
| 0 | $0.397, 0.397$ | $0.525, 0.302$ | $0.612, 0.239$ | $0.677, 0.194$ | $0.708, 0.172$ | $0.700, 0.187$ |

**Table 1: Equilibrium outputs $x_B^N\left(\bar{P}, \alpha\right), x_S^N\left(\bar{P}, \alpha\right)$**

Since $\bar{P}^* \approx 0.714$, $x_B^N\left(\bar{P}, \alpha\right)$ is increasing and $x_S^N\left(\bar{P}, \alpha\right)$ is decreasing in $\bar{P}$ if $\bar{P} \geq 0.72$ and $\alpha$ is sufficiently small. Next, we derive social welfare for the same pairs of $(\alpha, \bar{P})$ as **Table 1**.

| $\alpha \setminus \bar{P}$ | 0.9 | 0.85 | 0.8 | 0.75 | 0.72 | 0.7 |
|---|---|---|---|---|---|---|
| 1 | 0.629 | 0.645 | 0.655 | 0.661 | 0.664 | 0.666 |
| 3/4 | 0.629 | 0.644 | 0.653 | 0.658 | 0.660 | 0.662 |
| 1/2 | 0.629 | 0.642 | 0.646 | **0.647** | **0.646** | 0.649 |
| 1/4 | 0.629 | 0.638 | 0.635 | 0.629 | 0.624 | 0.629 |
| 0 | 0.629 | 0.632 | 0.620 | 0.603 | 0.592 | 0.600 |

**Table 2: Equilibrium Welfare**

We can confirm that social welfare is decreasing in $\bar{P}$ if $\alpha = 1$. However, if $\alpha = 0$, then social welfare is decreasing in $\bar{P}$ as long as $\bar{P} \in [0.72, 0.85]$. This is because $x_B^N\left(\bar{P}, 0\right) - x_S^N\left(\bar{P}, 0\right)$ is decreasing in $\bar{P}$ and $C'' > 0$. Moreover, even if $\alpha = 1/2$, the social welfare under $\bar{P} = 0.75$ is larger than that under $\bar{P} = 0.72$. Therefore, for some $\alpha$, making the price-cap level closer to the competitive level may be socially harmful. Finally, we consider the robustness of this result. In this example, the result is robust when we consider the average of welfare. That is, if each Cournot equilibrium is realized in the equal probability, then expected welfare can be decreased by some reduction of a price-cap level.

# 7 Concluding Remarks

This paper discusses the effect of a price-cap regulation on market outcomes. First, we consider the Cournot model. If the price-cap level is between the competitive price and the Cournot equilibrium price without any price-caps, then there are a unique symmetric equilibrium and asymmetric equilibria, and the price-cap is binding in any equilibria. Since we assume a convex cost function, the symmetric equilibrium is the most efficient one for any given price-cap levels. Thus, when we focus only on the symmetric equilibrium, any increase of the price-cap level is welfare improving as long as the level is less than the competitive price. However, when we focus on asymmetric equilibria, a decrease of the price-cap level may be socially harmful even if the level is more than the competitive price. To derive the result, we focus on the convex combinations of the symmetric equilibrium and the most asymmetric equilibrium. A coefficient of the convex combination represents the degree of symmetry of the equilibrium. Although Earle et al. (2007) also consider a Counot model, they focus only on the symmetric equilibria and show that social welfare is nonincreasing in the price cap level in the case where the demand is deterministic. On the other hand, we focus on the asymmetric equilibria and show that if the degree of symmetry is sufficiently low, then an increase in the price cap level may raise social welfare.

Moreover, we also derive the Stackelberg equilibrium under a price-cap regulation. If the price-cap level is binding, then the Stackelberg equilibrium is equivalent to the most asymmetric Cournot equilibrium. Therefore, social welfare can also be decreased by a reduction of the price-cap level.

We also consider the endogenous timing duopoly game where the firms engage in either Cournot or Stackelberg competition in the second stage depending on the actions of the firms in the first stage. We show that the Cournot competition happens as the pair of the weakly dominant strategies under any price-cap levels.

# Appendix

**Proof of Theorem 1.** We show the first sentence. First, suppose $\bar{P} \geq P(X^N)$. If $\bar{P}$ is not binding, then any pairs of outputs are not equilibrium unless $x_i = x^N$ for $i = 1, 2$. This is because $\left(x^N, x^N\right)$ is the pair of the unique Cournot equilibrium output without any regulation. Suppose that $(x_1, x_2) \neq \left(x^N, x^N\right)$ and $\bar{P}$ is binding; that is, $(x_1, x_2)$ satisfies $\bar{P} \leq P(x_1 + x_2)$. Then, $x_1 + x_2 \leq \bar{X}$ and $x_2 \leq \bar{X}/2 < x^N$. However, since $\varnothing(x_1 + x_2) \geq \varnothing(\bar{X}) \geq x^N$, firm 2 has the incentive to increase its output. Therefore, if $\bar{P} \leq P(X^N)$, then $\left(x^N, x^N\right)$ is the unique equilibrium.

Second, suppose $\bar{P} \in \left[P(X^N), P(X^C)\right]$. As is explained before, for given $\bar{P} \in \left(P(X^N), P(X^C)\right]$, $(x_1, x_2)$ is an equilibrium if and only if $x_1 + x_2 = \bar{X}$ and $\varnothing(\bar{X}) \leq x_i \leq \varnothing^C(\bar{X})$ for all $i = 1, 2$. First, suppose $\bar{X} - \varnothing(\bar{X}) < \varnothing^C(\bar{X})$. Then, $\bar{X} - \varnothing^C(\bar{X}) < \varnothing(\bar{X})$. If $x_1 > \bar{X} - \varnothing(\bar{X})$, then $\bar{X} - x_1 < \varnothing(\bar{X})$.

Therefore, if $x_1 > \bar{X} - \varnothing(\bar{X})$, then $(x_1, x_2)$ satisfying $x_1 + x_2 = \bar{X}$ is not equilibrium, because firm 2 has an incentive to increase its output. On the other hand, if $x_1 \leq \bar{X} - \varnothing(\bar{X})$, then $\bar{X} - x_1 \geq \varnothing(\bar{X})$. Thus, $(x_1, x_2)$ satisfying $x_1 + x_2 = \bar{X}$ is an equilibrium if and only if $\bar{X}/2 \leq x_1 \leq \bar{X} - \varnothing(\bar{X})$ or equivalently $\varnothing(\bar{X}) \leq x_2 \leq \bar{X}/2$. Second, suppose $\bar{X} - \varnothing(\bar{X}) > \varnothing^C(\bar{X})$. Then, $\bar{X} - \varnothing^C(\bar{X}) > \varnothing(\bar{X})$. If $x_1 > \varnothing^C(\bar{X})$, then $(x_1, x_2)$ is not equilibrium. On the other hand, if $x_1 \leq \varnothing^C(\bar{X})$, then $\bar{X} - x_1 \geq \bar{X} - \varnothing^C(\bar{X}) > \varnothing(\bar{X})$. Thus, $(x_1, x_2)$ satisfying $x_1 + x_2 = \bar{X}$ is an equilibrium if and only if $\bar{X}/2 \leq x_1 \leq \varnothing^C(\bar{X})$ or equivalently $\bar{X} - \varnothing^C(\bar{X}) \leq x_2 \leq \bar{X}/2$. These fact imply that $(x_1, x_2)$ is an equilibrium if and only if $x_1 = x_B^N\left(\bar{P}, \alpha\right) = \alpha\bar{X}/2 + (1-\alpha)\overline{x}^N\left(\bar{P}\right)$ and $x_2 = x_S^N\left(\bar{P}, \alpha\right) = \alpha\bar{X}/2 + (1-\alpha)\underline{x}^N\left(\bar{P}\right)$ for all $\alpha \in [0,1]$.

We show the second sentence. First, we show that $\underline{x}^N\left(\bar{P}\right)$ is a U-shaped function between $P(X^C)$ and $P(X^N)$. If $\bar{P} = P(X^C) + \varepsilon$ where $\varepsilon$ is a sufficiently small positive integer, then $\varnothing(\bar{X}) < \bar{X} - \varnothing^C(\bar{X})$ and thus $\underline{x}^N\left(\bar{P}\right) = \varnothing(\bar{X})$. This is because, if $\bar{P} = P(X^C)$, then $2\varnothing^C(\bar{X}) = \bar{X}$ and $\varnothing(\bar{X}) < \varnothing^C(\bar{X})$. Next, if $\bar{P} = P(X^N) - \varepsilon$, then $\bar{X} - \varnothing^C(\bar{X}) < \varnothing(\bar{X})$ and thus $\underline{x}^N\left(\bar{P}\right) = \bar{X} - \varnothing^C(\bar{X})$. This is because, if $\bar{P} = P(X^N)$, then $2\varnothing(\bar{X}) = \bar{X}$ and $\varnothing(\bar{X}) < \varnothing^C(\bar{X})$. Moreover, $\varnothing(\bar{X})$ is increasing and $\bar{X} - \varnothing^C(\bar{X})$ is decreasing in $\bar{X}$. Therefore, $\underline{x}^N\left(\bar{P}\right)$ is a U-shaped function between $P(X^C)$ and $P(X^N)$. Finally, we show $\underline{x}^N\left(\bar{P}\right) > 0$. To show $\varnothing(\bar{X}) > 0$ for all $\bar{X} \in \left[X^N, X^C\right]$ is sufficient for the proof. If $\varnothing(\bar{X}) = 0$, then $P(\bar{X}) - C'(0) \leq 0$. This contradicts $P(X^C) - C'(x^C) = 0$ and $x^C > 0$.

We show the third sentence. The minimizer of $\underline{x}^N\left(\bar{P}\right)$ between $P(X^C)$ and $P(X^N)$ is $\bar{P}$ that satisfies

$$\varnothing(\bar{X}) = \bar{X} - \varnothing^C(\bar{X}) \Leftrightarrow \varnothing(\bar{X}) + \varnothing^C(\bar{X}) = \bar{X}. \tag{4}$$

We similarly show that $\underline{x}^N\left(\bar{P}\right)$ is an inverted U-shaped function between $P(X^C)$ and $P(X^N)$. The maximizer of $\overline{x}^N\left(\bar{P}\right)$ between $P(X^C)$ and $P(X^N)$ is $\bar{P}$ satisfying (4). Therefore, $\bar{P}^*$ is also the maximizer of $\underline{x}^N\left(\bar{P}\right)$. **Q.E.D.**

**Proof of Corollary 2.** Suppose $\bar{P} \in \left(P(X^C), P(X^N)\right)$. Then, for $J = B, S$,

$$\pi_J^N\left(\bar{P}, \alpha\right) = \bar{P}x_J^N\left(\bar{P}, \alpha\right) - C(x_J^N\left(\bar{P}, \alpha\right))$$

and $\bar{P} \geq C'(x_B^N\left(\bar{P}, \alpha\right)) > C'(x_S^N\left(\bar{P}, \alpha\right))$ for all $\alpha \in [0,1)$. Therefore, we have the first result. Moreover, by the third result of Corollary 1 we have the second result. Next, we show that if $\partial x_J^N\left(\bar{P}, \alpha\right)/\partial\bar{P} > 0$, then $\pi_J^N\left(\bar{P}, \alpha\right)$. **Q.E.D.**

**Proof of Theorem 2.** The first sentence is obvious. We show the second sentence. Suppose $\bar{P} \in \left[P\left(X^C\right), P\left(X^S\right)\right] \Leftrightarrow \bar{X} \in \left[X^S, X^C\right]$. Since $\bar{X} \geq X^S$, $\bar{X} \geq x_L^S + r\left(x_L^S\right)$. Moreover, $r^C\left(x_L^S\right) > r\left(x_L^S\right)$. Therefore, if $x_L = x_L^S$, then $x_L^S + r_F\left(x_L^S, \bar{P}\right) \leq \bar{X}$ and thus the price-cap is binding.

First, we focus on $x_L$ satisfying

$$\bar{X} - x_L \leq r\left(x_L\right) \Leftrightarrow x_L + r\left(x_L\right) \geq \bar{X}. \tag{5}$$

Then, for any $x_L$ that satisfies (5), $r_F\left(x_L, \bar{P}\right) = r\left(x_L\right)$. Since $r' \in (-1, 0]$, $x_L + r\left(x_L\right)$ is increasing in $x_L$. Since $x_L^S + r\left(x_L^S\right) \leq \bar{X}$, $x_L \geq x_L^S$ for all $x_L$ that satisfies (5). By the second order condition of $L$, the profit of $L$ at $\hat{x}_L$ satisfying $\hat{x}_L + r(\hat{x}_L) = \bar{X}$ is higher than that at any $x_L$ satisfying (5).

Second, we focus on $x_L$ that satisfies

$$\bar{X} - x_L \geq r^C(x_L). \tag{6}$$

Then, for any $x_L$ that satisfies (6), $r_F\left(x_L, \bar{P}\right) = r^C(x_L)$. In this case, the price-cap must be binding. If $\bar{X} - x_L = r^C(x_L)$, then $x_L = \bar{X} - \varnothing^C(\bar{X})$. Since $r^{C\prime} \in (-1, 0]$, $x_L + r^C(x_L)$ is increasing in $x_L$. Thus, (6) implies $x_L \geq \bar{X} - \varnothing^C(\bar{X})$. Since $\varnothing^C(\bar{X}) \geq \bar{X} - \varnothing^C(\bar{X})$ and the second order condition is satisfied, the profit of $L$ at $x_L = \bar{X} - \varnothing^C(\bar{X})$ is higher than that at any $x_L > \bar{X} - \varnothing^C(\bar{X})$.

Third, we focus on $x_L$ that satisfies

$$\bar{X} - x_L \in \left[r(x_L), r^C(x_L)\right]. \tag{7}$$

Then, for any $x_L$ that satisfies (7), $r_F\left(x_L, \bar{P}\right) = \bar{X} - x_L$. Since (7) includes the case where $\bar{X} - x_L = r(x_L)$ and $\bar{X} - x_L = r^C(x_L)$, the profit maximizer of the leader satisfies (7). Since $x_L + r^C(x_L)$ is increasing in $x_L$, (7) is equivalent to $x_L \in \left[\bar{X} - \varnothing^C(\bar{X}), \bar{X} - \varnothing(\bar{X})\right]$ and the price-cap must be binding. Therefore, the profit of firm 1 is maximized at $\min\left\{\bar{X} - \varnothing(\bar{X}), \varnothing^C(\bar{X})\right\} = x_B^N\left(\bar{P}, 0\right)$ because $\varnothing^C(\bar{X}) > \bar{X} - \varnothing^C(\bar{X})$. Moreover,

$$x_F^S\left(\bar{P}\right) = \bar{X} - x_B^N\left(\bar{P}, 0\right) = \max\left\{\bar{X} - \varnothing^C(\bar{X}), \varnothing(\bar{X})\right\} = x_S^N\left(\bar{P}, 0\right).$$

In sum, if $\bar{P} \in \left[P\left(X^C\right), P\left(X^S\right)\right]$, then $\left(x_L^S\left(\bar{P}\right), x_F^S\left(\bar{P}\right)\right) = \left(x_B^N\left(\bar{P}, 0\right), x_S^N\left(\bar{P}, 0\right)\right)$. **Q.E.D.**

**Proof of Corollary 3.** We show $\bar{P}^* < P(X^S)$. Since $x_F^S = r\left(x_L^S\right)$, $x_F^S = \varnothing(X^S)$ and $x_L^S = X^S - \varnothing(X^S)$. Therefore, $\bar{P}^* < P(X^S)$. The remainder of the proof is direct from Corollary 1 and the fact that if $\bar{P} \in \left[P\left(X^C\right), P\left(X^S\right)\right]$, then $\left(x_L^S\left(\bar{P}\right), x_F^S\left(\bar{P}\right)\right) = \left(x_B^N\left(\bar{P}, 0\right), x_S^N\left(\bar{P}, 0\right)\right)$. **Q.E.D.**

**Proof of Theorem 3.** Amir and Grilo (1999, Theorem 2.2 and Corollary 2.5) show that $l$ is the dominant strategy of each firm. That is, $\pi_L^S\left(P(X^S)\right) \geq \pi_B^N\left(P(X^N), \alpha\right)$ and $\pi_B^N\left(P(X^N), \alpha\right) \geq \pi_F^S\left(P(X^S)\right)$ for all $\alpha \in [0, 1]$. Therefore, if $\bar{P} > P(X^N)$; that is, if $\bar{P}$ is not binding for any market structures, then Theorem 3 is obvious.

Next, we show that $l$ is a best response of a firm to $f$. First, suppose $\bar{P} \in \left(P(X^S), P(X^N)\right]$. In this case, $\bar{P}$ is binding in the Cournot competition but not binding in the Stackelberg competition. By Corollary 2, $\pi_L^S\left(\bar{P}\right) = \pi_B^N\left(P(X^S), 0\right) \geq \pi_B^N\left(P(X^S), \alpha\right)$ for all $\alpha \in [0, 1]$. Moreover, for any $\bar{P} \in \left(P(X^S), P(X^N)\right]$, $\pi_B^N\left(\bar{P}, 0\right) \geq \pi_B^N\left(\bar{P}, \alpha\right)$ for all $\alpha \in [0, 1]$. If a leader chooses its output $\bar{X} - \varnothing(\bar{X})$, then a follower chooes $\varnothing(\bar{X})$ and thus the profit of the leader is $\pi_B^N\left(\bar{P}, 0\right)$. Since $x_L^S = X^S - \varnothing(X^S)$ is the maximizer of the leader, $\pi_L^S\left(\bar{P}\right) \geq \pi_B^N\left(\bar{P}, 0\right)$. Second $\bar{P} \in \left[P(X^C), P(X^S)\right]$ By the definition of the Stackelberg equilibrium, $\pi_L^S\left(\bar{P}\right) \geq \pi_B^N\left(\bar{P}, 0\right)$ for all $\bar{P} \geq P(X^C)$ and $\alpha \in [0, 1]$.

Moreover, by Corollary 1, $\pi_B^N(\bar{P},0) \geq \pi_J^N(\bar{P},\alpha)$ for all $\alpha \in [0,1]$ and $J = B, S$. Therefore, $l$ is a best response of a firm to $f$.

Finally, we show that $l$ is a best response of a firm to $l$. First, suppose $\bar{P} \in \left(P(X^S), P(X^N)\right]$. By Corollary 2, $\pi_F^S(\bar{P}) = \pi_S^N\left(P(X^S), 0\right) \leq \pi_S^N\left(P(X^S), \alpha\right)$ for all $\alpha \in [0,1]$. Moreover, by Corollaries 1 and 3, $\partial x_S^N(\bar{P},0)/\partial\bar{P} > 0$. Therefore, $\pi_F^S(\bar{P}) \leq \pi_S^N(\bar{P},\alpha)$ for all $\alpha \in [0,1]$ and $\bar{P} \in \left(P(X^S), P(X^N)\right]$. Second, suppose $\bar{P} \in \left[P(X^C), P(X^S)\right]$. By Corollary 2, $\pi_F^S(\bar{P}) = \pi_S^N(\bar{P},0) \geq \pi_S^N(\bar{P},\alpha)$ for all $\alpha \in [0,1]$. Therefore, $l$ is a best response of a firm to $l$. **Q.E.D.**

**Proof of Proposition 1.** Since $C''(\cdot) > 0$, an expansion of the difference between $x_1$ and $x_2$ increases the industry cost and decreases social welfare if $x_1+x_2$ is not changed. If $\bar{P} \in \left(P(X^C), P(X^N)\right)$, then $x_B^N(\bar{P},\alpha) + x_S^N(\bar{P},\alpha) = \bar{X}$ for all $\alpha \in [0,1]$. Thus, $dW\left(x_B^N(\bar{P},\alpha), x_S^N(\bar{P},\alpha)\right)/d\alpha < 0$ for all $\alpha \in [0,1]$. **Q.E.D.**

**Proof of Proposition 4.** Fix $\alpha \in [0,1]$ and $\bar{P} \in \left[P(X^C), \bar{P}^*\right)$. Then,

$$\left(x_B^N(\bar{P},\alpha), x_S^N(\bar{P},\alpha)\right) = \left(\alpha\bar{X}/2 + (1-\alpha)\varnothing^C(\bar{X}), \alpha\bar{X}/2 + (1-\alpha)\left(\bar{X} - \varnothing^C(\bar{X})\right)\right)$$

and social welfare is

$$W\left(x_B^N(\bar{P},\alpha), x_S^N(\bar{P},\alpha)\right) =$$
$$\int_0^{\bar{X}} P(Z)dZ - C\left(\frac{\alpha\bar{X}}{2} + (1-\alpha)\varnothing^C(\bar{X})\right) - C\left(\frac{\alpha\bar{X}}{2} + (1-\alpha)\left(\bar{X} - \varnothing^C(\bar{X})\right)\right).$$

Since

$$\frac{dW\left(x_B^N(\bar{P},\alpha), x_S^N(\bar{P},\alpha)\right)}{d\bar{P}} = \frac{dW\left(x_B^N(\bar{P},\alpha), x_S^N(\bar{P},\alpha)\right)}{d\bar{X}}\frac{d\bar{X}}{d\bar{P}},$$

we will show that $dW\left(x_B^N(\bar{P},\alpha), x_S^N(\bar{P},\alpha)\right)/d\bar{X} > 0$ for all $\alpha \in [0,1]$. Moreover,

$$\begin{aligned}\frac{dW\left(x_B^N(\bar{P},\alpha), x_S^N(\bar{P},\alpha)\right)}{d\bar{X}} &= P(\bar{X}) - \left(\frac{\alpha}{2} + (1-\alpha)\varnothing^{C\prime}(\bar{X})\right)C'\left(\frac{\alpha\bar{X}}{2} + (1-\alpha)\varnothing^C(\bar{X})\right) \\ &- \left(1 - \frac{\alpha}{2} - (1-\alpha)\varnothing^{C\prime}(\bar{X})\right)C'\left(\left(1-\frac{\alpha}{2}\right)\bar{X} - (1-\alpha)\varnothing^C(\bar{X})\right).\end{aligned}$$

We have

$$P(\bar{X}) > C'\left(\frac{\alpha\bar{X}}{2} + (1-\alpha)\varnothing^C(\bar{X})\right) > C'\left(\frac{\alpha\bar{X}}{2} + (1-\alpha)\left(\bar{X} - \varnothing^C(\bar{X})\right)\right).$$

Moreover, since $\varnothing^{C\prime}(\bar{X}) < 0$, $\alpha/2 + (1-\alpha)\varnothing^{C\prime}(\bar{X}) < 1 - \alpha/2 - (1-\alpha)\varnothing^{C\prime}(\bar{X})$, $dW\left(x_B^N(\bar{P},\alpha), x_S^N(\bar{P},\alpha)\right)/d\bar{X} > 0$ for all $\alpha \in [0,1]$. **Q.E.D.**

# References


Acemoglu, D. and M. K. Jensen (2013) Aggregate Comparative Statics, Games and Economic Behavior 81, 27–49.

Amir, R. and I. Grilo (1999) Stackelberg versus Cournot Equilibrium, Games and Economic Behavior 26, 1-21.

Armstrong, M. and D. Sappington (2007) Recent Developments in the Theory of Regulation, In M. Armstrong and R. Porter (eds) Handboook of Industrial Organization. Vol.3, North Holland, Elsevier Publishing, Amsterdam.

Bhaskar, V. (1997) The Competitive Effects of Price-floors, Journal of Industrial Economics 45, 329-340.

Corchón, L. C. and F. Marcos (2012) Price Regulation in Oligopolistic Markets, International Scholarly Research Network ISRN Economics Article ID 509165.

van Damme, E. and S. Hurkens (1999) Endogenous Stackelberg Leadership, Games and Economic Behavior 28, 105–129

Earle, R., K. Schmedders, and T. Tatur (2007) On Price Caps Under Uncertainty, Review of Economic Studies 74, 93–111.

Etro, F. (2008) Stackelberg Competition with Endogenous Entry, Economic Journal 118, 1670-1697.

Grimm, V. and G. Zöttl (2010) Price Regulation under Demand Uncertainty, B.E. Journal of Theoretical Economics 10(1) (Advances), Article 26.

Hamilton, J. and S. Slutsky (1990) Endogenous Timing in Duopoly Games: Stackelberg or Cournot Equilibria. Games and Economic Behavior 2, 29-46.

Ino, H. and T. Matsumura (2012) How Many Firms Should Be Leaders? Beneficial Concentration Revisited. International Economic Review 53, 1323-1340.

Laffont, J. J. and J. Tirole (1994) A Theory of Incentives in Regulation and Procurement, MIT Press, Cambridge, MA.

Matsumura, T. (1995) Endogenous Timing in Multi-stage Duopoly Games, Japanese Economic Review 46: 257-265.

Matsumura, T. (2003) Stackelberg Mixed Duopoly with a Foreign Competitor, Bulletin of Economic Research 55(3): 275-287.

Matsumura, T. and A. Ogawa (2009) Payoff Dominance and Risk Dominance in the Observable Delay Game: A Note. Journal of Economics 97(3), 265-272

Matsumura, T. and Y. Okumura (2013) Privatization Neutrality Theorem Revisited, Economics Letters 118(2), 324-326.

Matsushima, N. and T. Matsumura (2003) Mixed Duopoly with Product Differentiation: Sequential Choice of Location, Australian Economic Papers 42(1), 18-34.

Milgrom, P. and J. Roberts (1994) Comparing Equilibria, American Economic Review 84(3) 441-459.

Reynolds, S. S. and D. Rietzke (2012) Price Caps, Oligopoly, and Entry. Mimeo

Roberts, J. and H. Sonnenschein (1976) On the existence of Cournot Equilibrium without Concave Profit Functions, Journal of Economic Theory 13(1) 112-117.

Roques, F. and N. Savva (2009) Investment under Uncertainty with Price Ceilings in Oligopolies, Journal of Economic Dynamics and Control, 33 (2), 507–524.

Sappington, D. E. M. (2002) Price Regulation, in M. Cave, S. Majumdar and I. Vogelsang (eds.), Handbook of Telecommunications Economics, North Holland, Elsevier Publishing, Amsterdam.

Vives, X. (2001) Oligopoly Pricing, MIT Press, Cambridge, MA.