Full Surplus Extraction and Costless Information Revelation in Dynamic Environments

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Abstract

In this paper, we study revenue maximizing mechanisms in dynamic environments. Especially we focus on the mechanisms which extract the full surplus. In order to extract the surplus, we have to (i) implement the efficient allocation rule, and (ii) retrieve whole the surplus and incentive payment. We show that intertemporal correlation between signals is crucial for both of these two requirements. Given the dynamic version of full rank condition proposed in this paper, we can construct a mechanism which is (within-period) ex post incentive compatible, individually rational, and extracts the whole surplus. Fortunately, this rank condition is weaker than the one in a static environment. Full surplus extraction might be possible even when state distributions are conditionally independent for all periods, or when we have "independent types" in the dynamic sense for most of periods.

Keywords: Dynamic mechanism design, perfect Bayesian equilibrium, revenue maximization, full surplus extraction.

1 Introduction

In this paper, we study revenue maximizing mechanisms under dynamic information environments with discrete state spaces. Especially, we focus on mechanisms which extract the full surplus. In order to maximize the revenue, the mechanism designer may have to implement an inefficient allocation to reduce the information rent. However, if the mechanism designer can implement the efficient allocation rule without leaving any information rent to the agents, it is definitely the best outcome for the mechanism designer who wants to maximize his revenue.

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Unfortunately, full surplus extraction is not always possible. For a static independent and private value auction problem, Myerson (1981) shows that the allocation of the revenue-maximizing mechanism is different from the efficient one. Furthermore, Myerson and Satterthwaite (1983) proves that the mechanism designer has to subsidize the agents in order to achieve the efficient allocation in bilateral trade problems. In these cases, the mechanism designer should leave information rent to give each agent an incentive to report his private information truthfully. When we have interdependent values, i.e., one agent's valuation for the object depends on the other agents' private signals, even the efficient allocation itself might not be implementable (for example, see Example 10.1 in Krishna (2002))[1]. Crémer and McLean (1985, 1988) shed light on the necessary and sufficient condition for full surplus extraction. They prove that in static problems with private values, if the agents' private signals are mutually correlated (satisfy the *full rank condition*), existence of a mechanism which extracts the whole surplus is guaranteed.

Should we give up full extraction when this condition fails, for example, agents' signal are distributed independently at the time of allocation problem? If we extend the scope of examination to dynamic environments, we still have a chance to extract the whole surplus. In order to get intuition for this, it is helpful to reconsider the meaning of "independent types." When all the stake holders of the allocation problem are truly blind to the *common shock factors*, agents' valuations must be correlated. For example, consider a used car auction. Consumers' reservation prices for the car is correlated if quality of the car has not been completely revealed. If the detail of the selling object, which is called common shock factor here, is uncovered, the remaining factors which determines each consumer's valuation, e.g. preference for color, size, design, etc. are private, so we can interpret this situation as a problem with independent types. Taken together, it is inevitable to leave information rent after all the common shocks are unveiled, however, if we can require agents to report their signals before it happens, it is possible to utilize the correlation lying there.

As a matter of fact, many real-world allocation problems are dynamic, and the mechanism designer may realize a desired outcome by making use of this dynamic nature. In dynamic information environments, each agent's state evolves over time, depending on the sequence of allocations and state profiles. In this case, even if the state distribution is independent conditional on the history, i.e., agents' signals does not have any correlation *within period*, one agent's today's signal might be correlated with another agent's tomorrow's signal. Naturally we may apply Crémer-McLean scheme here and reveal each agent's true state without leaving information rent.

Fortunately, the rank condition needed to construct a dynamic version of Crémer-McLean mechanism is substantially weaker than the one in static environments. In a classical one-shot allocation problem, its probabilistic structure is simple so that the only object which might be correlated with a agent' signal is the other agents' ones (in the same period). On the contrary, because of complexity of the dynamic information problem, there

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[1] Maskin (1992) shows that the single crossing condition iss sufficient to guarantee implementability of the efficient allocation.

are so many random variables we can make use of.

Thanks to the richness of the environment, we can discard intractable *intratemporal probabilistic structure*, which is the key thing of the static Crémer-McLean mechanism. By abandoning it, we can strengthen the incentive condition. As a matter of fact, in our mechanism, it is the strict best response to make a truthful report every period, regardless of the realization of the other agents' states. It is a quite desirable property because it implies that the mechanism (i) works well regardless of the intertemporal probabilistic structure, for example, under conditionally independent, short rank correlation or full rank correlation within period, and (ii) is robust to leakage of any information available up to that time. Even after discarding intratemporal structure and requiring a stronger incentive condition, we can construct a mechanism which extracts the whole surplus from *intertemporal correlation* under generic assumptions.

In order to represent this idea formally, we introduce the *costless revelation condition* of the state transition process. Intuitively, the costless revelation condition guarantees the possibility to construct a Crémer-McLean lottery which gives each agent (i) zero expected payoff if he reports the true state, and (ii) negative expected payoff if he misreports the state. Using this lottery, we can construct a payment rule which gives an incentive for truthful report, without leaving information rent.

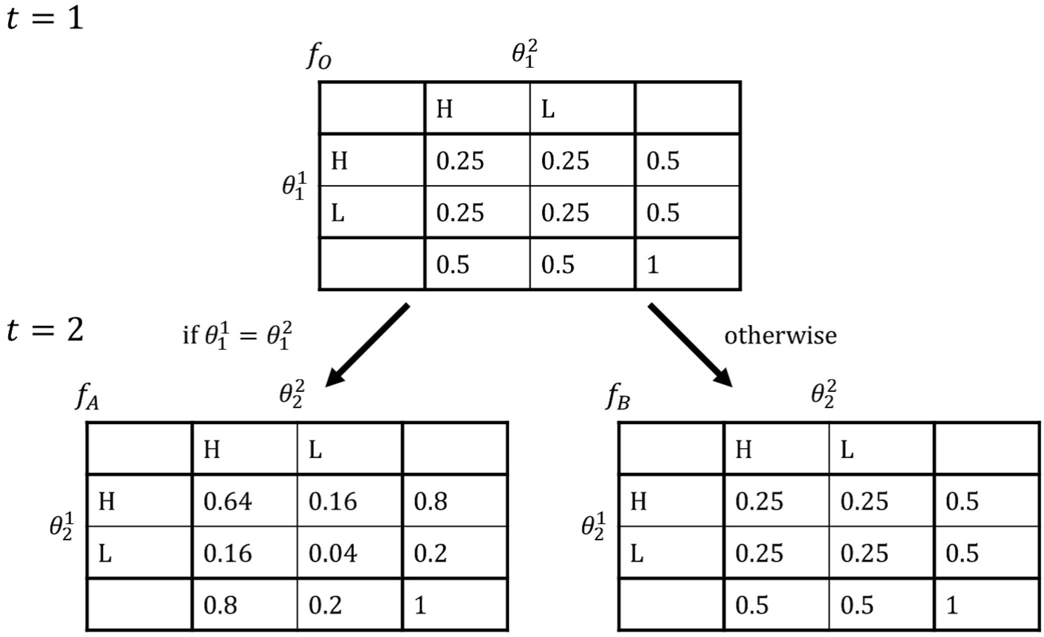
If the efficient allocation itself is implementable by the other scheme, the costless revelation condition *in the initial period* guarantees that the mechanism designer can retrieve whole the surplus and incentive payment, so it is sufficient for full surplus extraction. A number of papers have analyzed the design of efficient mechanisms. Bergemann and Välimäki (2010), Cavallo, Parkes, and Singh (2010) and Athey and Segal (2013) introduce several dynamic counterparts of the *Groves mechanism*, originally established by Groves (1973). If the precondition for their mechanisms (*private values*) are satisfied, we can implement efficient allocation by the Groves scheme and extract the surplus from the mechanism by costless revelation in the initial period.

If the costless revelation condition is satisfied throughout the time horizon, the state transition process also guarantees implementability of the efficient allocation. Together with the extraction scheme described in the previous paragraph, we can characterize a sufficient condition for the state transition to guarantee the possibility of full surplus extraction. Liu (2013) also analyzes the design of an efficient mechanism, which utilizes intertemporal correlation between state profiles. Despite Dasgupta and Maskin (2000) and Jehiel and Moldovanu (2001) prove that efficient allocation is not implementable if the problem is static and private signals are multidimensional and statistically independent, Liu shows that the efficient mechanism design is possible even when we have conditionally independent types, by utilizing the environment's dynamic feature. Although we limit our attention on environments with discrete and finite state spaces, our costless revelation condition is further weaker than Liu's identification condition, because we utilize intertemporal correlation not only between two adjacent, but longer consecutive periods.

Though our rank condition is satisfied generically as long as multiple agents participate in the mechanism, when it fails, full surplus extraction may not be possible and some different techniques are needed to maximize the revenue. Baron and Besanko (1984) and Courty and Li (2000) considere optimal pricing policies in a two-period problem. Battaglini (2005) studies the optimal contract in a model with infinite time horizon when the preference of the buyer follows a Markov process. All of the papers above analyzed single-agent models, so our Crémer-McLean scheme is not applicable. In contrast, Pavan, Segal, and Toikka (2014) studies a general dynamic information model with multi-agent with private and independent values in the dynamic sense. They establish a dynamic version of revenue equivalence principle and derive the optimal revenue-maximizing mechanism. When the rank condition fails, their works give us an insight for methods to maximize the revenue.

2 Short-period examples

2.1 A two-period example



	H	L	
H	0.25	0.25	0.5
L	0.25	0.25	0.5
	0.5	0.5	1

	H	L	
H	0.64	0.16	0.8
L	0.16	0.04	0.2
	0.8	0.2	1

	H	L	
H	0.25	0.25	0.5
L	0.25	0.25	0.5
	0.5	0.5	1

Figure 1: A two-period example

Consider a two-period example. Two identical agents (say, agent 1 and 2) participate in the mechanism. Each agent's outside opportunity is fixed to zero. The game proceeds in a following way.

Period 1 Agent i observes his private state $\theta_1^i \in \{H, L\}$. The joint distribution of $\theta_1 = (\theta_1^1, \theta_1^2)$ is given by f_O (see Figure 1). Each agent reports θ_t^i, which is not necessarily the truth, and the mechanism designer determines the allocation in period 1, $x_t \in X = \{\chi(H,H), \chi(H,L), \chi(L,H), \chi(L,L)\}$.

Period 2 Basically, the procedure is same as that in period 1, however, the state distribution is different. If $\theta_1^1 \neq \theta_1^2$, θ_2 is drawn from f_A. Otherwise, it is drawn from f_B. After the report and allocation, each agent's payoff is finalized. Agent i's valuation is given by $v(x_1, \theta_1^i) + v(x_2, \theta_2^i) + y^i$, where v is the agent's valuation function and y^i is monetary transfer.

Assume that $\chi(\theta_t)$ maximizes the social welfare in period t, when the social state is θ_t [2]. The mechanism designer wants to maximize ex ante revenue from this mechanism and can commit a mechanism before the game, and to achieve the efficient outcome, he chooses $\chi(\theta_t)$ in period t if the reported state profile is θ_t. Since we have private values here, the efficient allocation is ex post implementable by the Groves mechanism, whose payment rule is given by $\psi^i(\theta_1, \theta_2) = v(\chi(\theta_1), \theta_1^{-i}) + v(\chi(\theta_2), \theta_2^{-i})$. On the contrary, since the state distributions are conditionally independent, it is impossible to extract the full surplus if we apply static mechanisms periodically.

However, making use of the problem's dynamic structure, the mechanism designer can extract the whole surplus. Consider a payment rule w^i s.t.

$$w^i(\theta_1, \theta_2^{-i}) = \begin{cases} 1 & (\text{if } \theta_1 \in \{(H,H),(L,L)\} \text{ and } \theta_2^{-i} = H) \\ -4 & (\text{if } \theta_1 \in \{(H,H),(L,L)\} \text{ and } \theta_2^{-i} = L) \\ -1 & (\text{if } \theta_1 \in \{(H,L),(L,H)\} \text{ and } \theta_2^{-i} = H) \\ 1 & (\text{if } \theta_1 \in \{(H,L),(L,H)\} \text{ and } \theta_2^{-i} = L) \end{cases} \tag{1}$$

Then, for any realization of θ_1, $\mathbb{E}\left[w^i((\hat{\theta}_1^i, \theta_1^{-i}), \tilde{\theta}_2^{-i}) \middle| \theta_1 \right] = 0$ if $\hat{\theta}_1^i = \theta_1^i$ and otherwise, $\mathbb{E}\left[w^i((\hat{\theta}_1^i, \theta_1^{-i}), \tilde{\theta}_2^{-i}) \middle| \theta_1 \right] < 0$. To see this, suppose that $\theta_1 = (H, H)$ and agent 2 reports his state truthfully. Since $\theta_1^1 = \theta_1^2 = H$, θ_2 is drawn from f_A, so $\Pr(\tilde{\theta}_2^2 = H) = 0.8$ and $\Pr(\tilde{\theta}_2^2 = L) = 0.2$. In this case, if agent 1 reports truthfully, his expected payment is $0.8 \cdot 1 + 0.2 \cdot (-4) = 0$. Conversely, if he misreports his state, $0.8 \cdot (-1) + 0.2 \cdot 1 = -0.6 < 0$. Remaining parts can be verified in a similar way.

Let the payment rule ψ^i be

$$\begin{aligned} \psi^i(\theta_1, \theta_2) = {} & v(\chi(\theta_2), \theta_2^{-i}) - v(\chi(\theta_1), \theta_1^i) \\ & - \mathbb{E}\left[v(\chi(\tilde{\theta}_2), \tilde{\theta}_2^i) + v(\chi(\tilde{\theta}_2), \tilde{\theta}_2^{-i}) \middle| \theta_1 \right] + \alpha \cdot w^i(\theta_1, \theta_2^{-i}). \end{aligned} \tag{2}$$

Then, in period 2, each agent tell the truth regardless of θ_2^{-i} since only the first term depends on $\hat{\theta}_2^i$. Consider agent i's incentive compatibility in period 1. Given θ_1, agent i's utility maximization problem in period 1 is

$$\max_{\hat{\theta}_1^i \in \{H,L\}} \left\{ v(\chi(\hat{\theta}_1^i, \theta_1^{-i}), \theta_1^i) + \mathbb{E}\left[v(\chi(\tilde{\theta}_2), \tilde{\theta}_2^i) + \psi^i((\hat{\theta}_1^i, \theta_1^{-i}), \tilde{\theta}_2) \middle| \theta_1 \right] \right\} \tag{3}$$

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[2] Formally, $\chi(\theta_t) \in \arg\max_{x_t} [v^1(x_t, \theta_t^1) + v^2(x_t, \theta_t^2)]$. Since the allocation does not affect the state transition in this example, we don't have to worry about continuation values.

Expanding ψ^i in (3), the objective function can be rewritten as

$$\begin{aligned} & v(\chi(\hat{\theta}_1^i, \theta_1^{-i}), \theta_1^i) - v(\chi(\hat{\theta}_1^i, \theta_1^{-i}), \hat{\theta}_1^i) \\ & + \mathbb{E}\left[v(\chi(\tilde{\theta}_2), \tilde{\theta}_2^i) + v(\chi(\tilde{\theta}_2), \tilde{\theta}_2^{-i}) \middle| \, \theta_1 \right] - \mathbb{E}\left[v(\chi(\tilde{\theta}_2), \tilde{\theta}_2^i) + v(\chi(\tilde{\theta}_2), \tilde{\theta}_2^{-i}) \middle| \, (\hat{\theta}_1^i, \theta_1^{-i}) \right] \\ & + \alpha \cdot \mathbb{E}\left[w^i((\hat{\theta}_1^i, \theta_1^{-i}), \tilde{\theta}_2^{-i}) \middle| \, \theta_1 \right]. \end{aligned} \tag{4}$$

Therefore, if agent i tell the truth in period 1, his expected payoff from the mechanism is zero. In addition, letting α sufficiently large, we can make the value of objective function in (3).

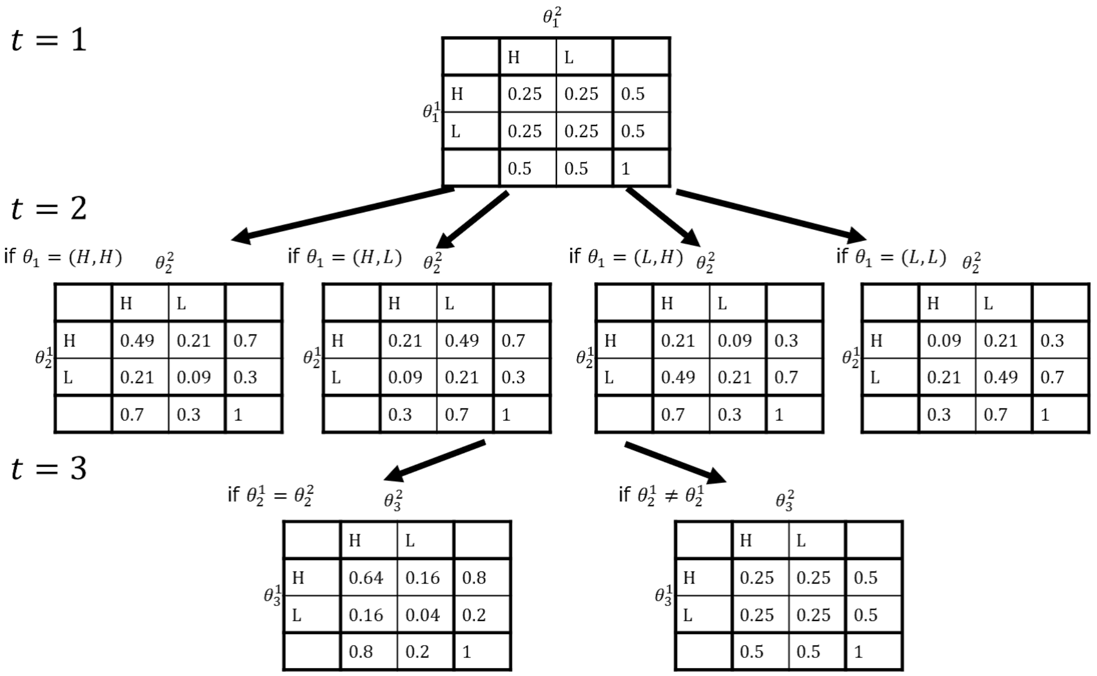
Finally, we consider individual rationality. In period 1, each agent's on path expected payoff is zero, so it is equal to the value of outside opportunity here. On the contrary, in period 2, both agents do not always want to attend the mechanism. However, this is not a very serious problem since by requiring each agent to make a deposit in period 1 and returning it in period 2, we can keep participation constraints in period 2, without hurting any incentive structure.

The mechanism we constructed above is (i) efficient, and (ii) each agent's individual rationality is satisfied with equality. Therefore, the mechanism designer's ex ante revenue attains the maximum amount. Here are three remarks. First, as we will show later, such a lottery w^i exists as long as given for any θ_1^{-i}, θ_1^i and θ_2^{-i} have a full-rank correlation. In this example, given $\theta_1^2 = H$, the probability vectors of $\tilde{\theta}_2^2$ are $(0.8, 0.2)$ given $\theta_1^1 = H$ and $(0.5, 0.5)$ given $\theta_1^1 = L$, so they are linearly independent. Second, when we consider the possibility of full surplus extraction in dynamic environments, it is effective to divide it to two different problems—implementation and extraction. In this case, we used the Groves scheme to implement the efficient allocation in period 2, and extract the expected surplus and subsidy conditional on θ_1 using the Crémer-McLean scheme. In general, if we can reveal θ_1 without leaving information rent, the mechanism designer can extract the surplus from the mechanism. Finally, although we considered a conditionally independent type environment here, the logic we introduced here is free from the intratemporal probabilistic structure. In every period, each agent has an incentive to report his state truthfully, regardless of the realization of the opponent's private signal. Therefore, this technique is applicable to the environment where agents' signals are intratemporally correlated.

2.2 A three-period example

In order to gain further insights, let us introduce one more example. The game procedure is basically the same as the previous example, while the new example has three periods. The state transition is depicted in Figure 2. We add new period in the middle of the two-period example, and in this new period (period 2), each agent's marginal distribution depends only on his own past state. Each agent' state in period 2 is persistent, so θ_2^i is likely to be 'high' (marginal probability vector is $(0.7, 0.3)$) if $\theta_1^i = H$ and vice versa.

When we focus on period 1 and 2, each agent has *independent types*, which is defined by Bergemann and Välimäki (2010) and Athey and Segal (2013). Each agent's private states



$t = 1$: θ_1^1 (rows), θ_1^2 (columns)

	H	L	
H	0.25	0.25	0.5
L	0.25	0.25	0.5
	0.5	0.5	1

$t = 2$:

if $\theta_1 = (H,H)$: θ_2^1, θ_2^2

	H	L	
H	0.49	0.21	0.7
L	0.21	0.09	0.3
	0.7	0.3	1

if $\theta_1 = (H,L)$: θ_2^1, θ_2^2

	H	L	
H	0.21	0.49	0.7
L	0.09	0.21	0.3
	0.3	0.7	1

if $\theta_1 = (L,H)$: θ_2^1, θ_2^2

	H	L	
H	0.21	0.09	0.3
L	0.49	0.21	0.7
	0.7	0.3	1

if $\theta_1 = (L,L)$: θ_2^1, θ_2^2

	H	L	
H	0.09	0.21	0.3
L	0.21	0.49	0.7
	0.3	0.7	1

$t = 3$:

if $\theta_2^1 = \theta_2^2$: θ_3^1, θ_3^2

	H	L	
H	0.64	0.16	0.8
L	0.16	0.04	0.2
	0.8	0.2	1

if $\theta_2^1 \neq \theta_2^1$: θ_3^1, θ_3^2

	H	L	
H	0.25	0.25	0.5
L	0.25	0.25	0.5
	0.5	0.5	1

Figure 2: A three-period example

are not only conditionally independent but also intertemporally independent, i.e., θ_1^i is not informative to predict θ_2^{-i}.

Although we cannot directly apply the intertemporal Crémer-McLean scheme introduced in the previous example, we still have a chance to extract the whole surplus. While θ_1^i and θ_2^{-i} are not correlated, there is positive correlation between θ_1^i and θ_2^i. Of course, if we construct a Crémer-McLean lottery from them, each agent sometimes has an incentive to tell a lie in period 2. However, even then, giving a further stronger incentive to tell the truth in period 2 by a different scheme, we can make agent 2 report his state truthfully in period 2. Given this, it is possible to reveal agent i's state in period 1 without leaving information rent.

Define w_1^i and w_2^i for $i = 1, 2$ as follows.

$$w_1^i(\theta_1^i, \theta_2^i) = \begin{cases} 1 & (\text{if } \theta_1^i = \theta_2^i) \\ -1 & (\text{if } \theta_1^i \neq \theta_2^i) \end{cases} \tag{5}$$

and

$$w_2^i(\theta_2, \theta_3^{-i}) = \begin{cases} 1 & (\text{if } \theta_2 \in \{(H,H),(L,L)\} \text{ and } \theta_3^{-i} = H) \\ -4 & (\text{if } \theta_2 \in \{(H,H),(L,L)\} \text{ and } \theta_3^{-i} = L) \\ -1 & (\text{if } \theta_2 \in \{(H,L),(L,H)\} \text{ and } \theta_3^{-i} = H) \\ 1 & (\text{if } \theta_2 \in \{(H,L),(L,H)\} \text{ and } \theta_3^{-i} = L) \end{cases} \tag{6}$$

We can easily verify that $\mathbb{E}\left[w_1^i(\hat{\theta}_1^i, \tilde{\theta}_2^{-i}) \middle| \theta_1\right] = 0$ if $\hat{\theta}_1^i = \theta_1^i$ and $\mathbb{E}\left[w_1^i(\hat{\theta}_1^i, \tilde{\theta}_2^i) \middle| \theta_1\right] < 0$

otherwise, and $\mathbb{E}\left[w_2^i((\hat{\theta}_2^i,\theta_2^{-i}),\tilde{\theta}_3^{-i})\Big|\,\theta_2\right]=0$ if $\hat{\theta}_2^i=\theta_2^i$ and $\mathbb{E}\left[w_2^i((\hat{\theta}_2^i,\theta_2^{-i}),\tilde{\theta}_3^{-i})\Big|\,\theta_2\right]<0$ otherwise. Define the payment rule as

$$\begin{aligned}\psi^i(\theta_1,\theta_2,\theta_3)=&-v(\chi(\theta_1),\theta_1^i)-v(\chi(\theta_2),\theta_2^i)\\&+v(\chi(\theta_3),\theta_3^{-i})-\mathbb{E}\left[v(\chi(\tilde{\theta}_3),\tilde{\theta}_3^i)+v(\chi(\tilde{\theta}_3),\tilde{\theta}_3^{-i})\Big|\,\theta_2\right]\\&+\alpha_1 w_1^i(\theta_1^i,\theta_2^i)+\alpha_2 w_2^i(\theta_2,\theta_3^{-i})\end{aligned}\tag{7}$$

Again, we check the agent i's utility maximization problem backward. Since $\hat{\theta}_3^i$ affect only the allocation and $v(\chi(\theta_3),\theta_3^{-i})$, it is just a static Groves mechanism, so truthtelling is incentive compatible regardless of the past history.

Next, we consider agent i's problem in period 3. Given that this agent will make a truthful report in period 3, it can be written as

$$\max_{\hat{\theta}_1^i\in\{H,L\}}\left\{\begin{aligned}&v(\chi(\hat{\theta}_2^i,\theta_2^{-i}),\theta_2^i)-v(\chi(\hat{\theta}_2^i,\theta_2^{-i}),\hat{\theta}_2^i)\\&+\mathbb{E}\left[v(\chi(\tilde{\theta}_3),\tilde{\theta}_3^i)+v(\chi(\tilde{\theta}_3),\tilde{\theta}_3^{-i})\Big|\,\theta_2\right]-\mathbb{E}\left[v(\chi(\tilde{\theta}_3),\tilde{\theta}_3^i)+v(\chi(\tilde{\theta}_3),\tilde{\theta}_3^{-i})\Big|\,(\hat{\theta}_2^i,\theta_2^{-i})\right]\\&+\alpha_1 w_1^i(\theta_1^i,\hat{\theta}_2^i)+\alpha_2\mathbb{E}\left[w_2^i((\hat{\theta}_2^i,\theta_2^{-i}),\theta_3^{-i})\Big|\,\theta_2\right]\end{aligned}\right\}.\tag{8}$$

Note that "θ_1^i" appears here is not necessarily the true state but the reported i's state in period 1. If agent i tell the truth, his expected payoff is $\alpha_1 w_1^i(\theta_1^i,\theta_2^i)$. Therefore, by setting α_2 sufficiently large (relative to α_1), we can prevent each agent from deviation, since $\mathbb{E}\left[w_2^i((\hat{\theta}_2^i,\theta_2^{-i}),\theta_3^{-i})\Big|\,\theta_2\right]<0$ if $\hat{\theta}_2^i\neq\theta_2^i$ and all the other terms are bounded.

Given that each agent does not have an incentive to misreport his state *regardless of the history*, the problem in period 1 is completely isomorphic to the previous example. Sufficiently large α_1 gives each agent an incentive for truthful report, and each agent's expected payoff from the mechanism is fixed to zero. Participation constraints in later periods are easily resolved by requiring a deposit in period 1, and returning it in the final period. Thus, this mechanism extracts the whole surplus.

As we will show later, we can apply this scheme to longer-period problems. Surprisingly, it is verified that we can extract the full surplus even when agents' states are distributed independently (in the dynamic sense) for most of periods, as long as there exists a time point when the private information is revealed costlessly.

3 Set up

Consider an environment with finite agents and discrete-time, infinite time horizon, indexed by $i\in\mathcal{I}=\{1,2,...,I\}$ and $t\in\mathbb{N}=\{0,1,2,...\}$ respectively. In each period t, each agent i observes the realization of his private state $\theta_t^i\in\Theta_t^i$ and the public state $\theta_t^0\in\Theta_t^0$. We assume that Θ_t^i is discrete and finite for all i and t. $\Theta_t=\times_{i=0}^I\Theta_t^i$ denotes the state space in t. After the state θ_t is realized, the mechanism designer decides an allocation $x_t\in X_t$ and a transfer $y_t=(y_t^1,y_t^2,...,y_t^I)\in\mathbb{R}^I$ according to the mechanism he committed ex ante.

Agent i wants to maximize his payoff,

$$\sum_{t=0}^{\infty} \delta^t [v_t^i(x_t, \theta_t) + y_t^i] \tag{9}$$

determined by the sequences of states $(\theta_t)_{t=0}^{\infty}$, allocations $(x_t)_{t=0}^{\infty}$ and transfers $(y_t)_{t=0}^{\infty}$, where $\delta \in (0,1)$ [3] is a discount factor and $v_t^i : X_t \times \Theta_t \to \mathbb{R}$ is agent i's valuation function in t. We assume that $(v_t^i)_{t=0}^{\infty}$ is bounded uniformly.

The state distribution in the initial period is given by $\mu_0 \in \Delta(\Theta_0)$ and subsequent states are distributed according to the transition probability measure $\mu_t : X_{t-1} \times \Theta_{t-1} \to \Delta(\Theta_t)$, which is measurable. As Athey and Segal (2013) noted, Markov formulation is without loss of generality since we can take the state space which includes all the past history. We call $(\Theta_t, \mu_t)_{t=0}^{\infty}$ the *information structure.*

Here we focus on direct mechanisms. In each period t, each agent i reports his state θ_t^i. Then the mechanism designer decides x_t and y_t, according to the mechanism (χ, ψ), where $\chi_t : \Theta_t \to X_t$ is the allocation rule in t and $\psi_t = (\psi_t^1, ..., \psi_t^I)$ where $\psi^i : \Theta_0 \times \cdots \times \Theta_t \to \mathbb{R}$ is a payment rule of agent i in t. χ and ψ are sequences of these payment rules, i.e., $\chi := (\chi_t)_{t=0}^{\infty}$ and $\psi := (\psi_t)_{t=0}^{\infty}$. Note that although we can restrict ψ_t^i's domain to Θ_t without loss of generality because the public state θ_t^0 can include all the past reported state profiles, we explicitly state here because the payment rules proposed in this paper crucially depend on the history of reports. $\boldsymbol{\theta}_\tau^t$ denotes the sequence of reported state profiles from τ to t, i.e., $\boldsymbol{\theta}_\tau^t := (\theta_\tau, \theta_{\tau+1}, \cdots, \theta_t)$. Especially, we define $\boldsymbol{\theta}^t := \boldsymbol{\theta}_0^t$. We call $(\Theta_t, \mu_t, \chi_t)_{t=0}^{\infty}$ the *state transition process.*

In this setting, each agent i chooses a reporting strategy $\sigma_t^i : \Theta_t^0 \times \Theta_t^i \to \Theta_t^i$ which maximizes his expected present value (hereafter we use EPV as an abbreviation) in each period t. The *truthful strategy* of agent i always reports his current state θ_t^i, i.e., $\sigma_t^i(\theta_t^0, \theta_t^i) = \theta_t^i$ for all θ_t^0 and θ_t^i. A mechanism $M = (\chi, \psi)$ is *interim incentive compatible* if the truthful strategy profile with the consistent belief is a Perfect Bayesian Equilibrium.

We focus on a stronger equilibrium concept in this paper. We say that (χ, ψ) is *within-period ex post incentive compatible* (wp-EPIC) if truthtelling is a best response regardless of the history and *the current state of the other agents.* This definition is the same as the one in Bergemann and Välimäki (2010) and Athey and Segal (2013).

Define $V_t^i(\chi) : \Theta_t \to \mathbb{R}$ as agent i's on-path periodic ex post expected present value (EPV) from valuations, given that the current state profile is θ_t, and the mechanism designer adopts the allocation rule χ. Formally, $V_t^i(\chi)$ is defined as

$$V_t^i(\chi)(\theta_t) := \mathbb{E}\left[\sum_{\tau=t}^{\infty} \delta^\tau [v_t^i(\chi_\tau(\tilde{\theta}_\tau), \tilde{\theta}_\tau)] \middle| \theta_t \right], \tag{10}$$

where expectations are taken according to $(\Theta_t, \mu_t, \chi_t)_{t=0}^{\infty}$. Clearly, this does not depend on payment rule ψ. Since monetary transfers are zero-sum, social surplus is determined only by the EPV from valuations. Given an allocation policy χ, *expected social surplus*

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[3] When we consider a finite-horizon problem, $\delta = 1$ is permitted.

is $W(\chi) := \mathbb{E}\left[\sum_{i\in\mathcal{I}} V_0^i(\chi)(\tilde{\theta}_0)\right]$. An *efficient allocation policy* is an allocation policy χ^* which maximizes expected social surplus $W(\chi)$. For notational simplicity, in this paper we abbreviate χ if it does not cause confusion. In most cases, χ denotes the efficient allocation policy. To be precise, there may exist several different efficient allocation rules, since different allocation rules sometimes give the same social surplus. For simplicity, we ignore this multiplicity by assuming that tie is broken in some arbitrary, but deterministic ways.

Even after we pick an efficient allocation rule, there are tremendously many payment rules which implement it. Similar to the EPV from valuation V_t^i, given an allocation rule χ and a payment rule ψ, here we define agent i's EPV from monetary transfer as

$$\Psi_t^i(\boldsymbol{\theta}^t) := \mathbb{E}\left[\sum_{\tau=t}^{\infty} \delta^\tau [\psi_t^i(\tilde{\boldsymbol{\theta}}^\tau)] \middle| \boldsymbol{\theta}^t \right]. \tag{11}$$

Here we also abbreviate χ for notational simplicity.

Using these EPV terms, each agent i's on-path EPV in period t with the state θ_t can be written as

$$U_t^i(\boldsymbol{\theta}^t) := V_t^i(\theta_t) + \Psi_t^i(\boldsymbol{\theta}^t). \tag{12}$$

Note that thanks to the Markov formulation of the state transition, the history until $t-1$ affect only EPV of the payment. Furthermore, if the agents have beliefs consistent with a truthful equilibrium, i.e., assign probability 1 on the other agents' latest reports, $\mathbb{E}[f(\tilde{\boldsymbol{\theta}}_{t+1}^\infty)|x_t,\theta_t] = \mathbb{E}[f(\tilde{\boldsymbol{\theta}}_{t+1}^\infty)|(x_\tau)_{\tau=0}^t, \boldsymbol{\theta}^t]$ always holds.

Using this notation, it is easy to show that the mechanism (χ,ψ) is wp-EPIC if and only if for all $i\in\mathcal{I}$, $t\in\mathbb{N}$, today's true state profile $\theta_t\in\Theta_t$, possible report $\hat{\theta}_t^i\in\Theta_t^i$, and the history of reported state profile $\boldsymbol{\theta}^{t-1}$,

$$\begin{aligned} U_t^i(\boldsymbol{\theta}^t) \geq\, & v_t^i(\chi(\hat{\theta}_t^i,\theta_t^{-i}),\theta_t) + \psi_t^i(\boldsymbol{\theta}^{t-1},(\hat{\theta}_t^i,\theta_t^{-i})) \\ & + \delta\mathbb{E}\left[U_t^i(\boldsymbol{\theta}^{t-1},(\hat{\theta}_t^i,\theta_t^{-i}),\tilde{\theta}_{t+1}) \middle| \chi_t(\hat{\theta}_t^i,\theta_t^{-i}),\theta_t\right], \end{aligned} \tag{13}$$

i.e., one-shot deviation is not profitable after any history.

We define within period ex post individual rationality as follows.

Definition 1 (wp-EPIR) A mechanism (χ,ψ) is *within period ex post individually rational* (wp-EPIR) if for all $i,t,$, and $\boldsymbol{\theta}^t$,

$$U_t^i(\boldsymbol{\theta}^t) \geq O_t^i(\theta_t), \tag{14}$$

where $O_t^i : \Theta^t \to \mathbb{R}$ denotes the value of agent i's outside opportunity (assumed to be uniformly bounded).

Intuitively, $O_t^i(\theta_t)$ is the value agent i can acquire without permission or cooperation of the mechanism designer and the other agents. The mechanism designer cannot prevent him from taking this option. However, if we take the possible allocation space X_t sufficiently rich, we can include this "outside option behavior" as a part of allocations. The mechanism designer can designate this kind of allocations as long as each agent's EPV is equal to or larger than his outside opportunity. Otherwise, the agent escapes from the mechanism and does everything by himself.

This definition of individual rationality is most flexible since the value of agent i's outside opportunity can depend on all the information available for him at that moment. O_t^i actually changes depending on the state in some problems. For example, considering public goods provision, "exit" means leaving from the meeting and stopping payment on the park. In this case, even after this agent decides to exit from the mechanism, he can enjoy the park built by the other agents because neighbors cannot exclude him. Furthermore, expected quality of the park depends on the amount of neighbors' investment, so it crucially depends on the social state.

4 Surplus Extraction

As I noted it in the introduction, we will investigate mechanisms which extract whole the surplus from the agents. To consider this problem, firstly we give a definition of the mechanism designer's 'revenue' in dynamic environments.

Definition 2 (Revenue) The *revenue* R of a dynamic mechanism (χ, ψ) is defined as

$$R := -\mathbb{E}\left[\sum_{i\in\mathcal{I}} \Psi_0^i(\tilde{\theta}_0)\right]. \tag{15}$$

In other words, R is the summation of ex ante EPV of payments from the agents. We can interpret this as the mechanism designer's expected monetary payoff from the mechanism (χ, ψ). Since the mechanism designer commits a dynamic mechanism ex ante, it is natural to assume that he wants to maximize ex ante payoff. Furthermore, if the mechanism designer's utility is linear in monetary transfer (risk-neutral) and he shares the same discount factor δ with the agents, his payoff is ex ante payoff is exactly same as the 'revenue' defined here.

Next, we will formally define "full surplus extraction" under the dynamic environment.

Definition 3 (Full Surplus Extraction) A mechanism $M = (\chi, \psi)$ *extracts the full surplus* if (i) the allocation rule χ is efficient and (ii) the mechanism does not leave any information rent, i.e.,

$$R = \mathbb{E}\left[\sum_{i\in\mathcal{I}} [V_0^i(\tilde{\theta}_0) - O_0^i(\tilde{\theta}_0)]\right] \tag{16}$$

holds.

Clearly, this amount of revenue is maximal since the allocation is efficient and wp-EPIR in the initial period requires us that $-\Psi_0^i(\theta_0) \leq V_0^i(\theta_0) - O_0^i(\theta_0)$ for all i and θ_0. Therefore, it is the most desirable outcome for the mechanism designer who wants to maximize the revenue from the mechanism, if it is possible.

Then, what is the sufficient condition for the existence of a mechanism which achieves full surplus extraction? To address this problem, it is effective to "divide each difficulties as many parts as is feasible and necessary to resolve it." In fact, there are three difficulties; (i) implement the efficient allocation (meet the incentive compatibility), and (ii) reveal the initial state without leaving information rent (make the individual rationality binds in the initial period).

In this section, we hire one of existing schemes to implement the efficient allocation, and retrieve the surplus and incentive payments by adding a *participation fee* to the original mechanism. Here the possibility of full surplus extraction is equivalent to the existence of such a participation fee schedule.

In order to address the points at hand, we introduce one more concept.

Definition 4 (Costless Revelation) An state transition process $(\Theta_\tau, \mu_\tau, \chi_\tau)_{\tau=0}^\infty$ satisfies the *costless revelation condition in* t if there exists a (part of) payment rule $\phi = ((\phi_\tau^i)_{i\in\mathcal{I}})_{\tau=t+1}^\infty$ s.t.

(i) each ϕ_τ^i is uniformly bounded and irrelevant to the history of reported state profiles until $t-1$, i.e., $\phi_\tau^i : \times_{s=t}^\tau \Theta_s \to \mathbb{R}$ for all i and τ,

(ii) Φ_t^i satisfies

$$\mathbb{E}\left[\Phi_{t+1}^i(\theta_t, \tilde{\theta}_{t+1}) \middle| \chi_t(\theta_t), \theta_t\right] = 0 \tag{17}$$

and

$$\mathbb{E}\left[\Phi_{t+1}^i((\hat{\theta}_t^i, \theta_t^{-i}), \tilde{\theta}_{t+1}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t\right] < 0 \tag{18}$$

for all $i \in \mathcal{I}$, $\theta_t \in \Theta_t$ and $\hat{\theta}_t^i \neq \theta_t^i$,

(iii) Φ_τ^i satisfies

$$\Phi_\tau^i(\boldsymbol{\theta}_t^\tau) \geq \phi_t^i(\boldsymbol{\theta}_t^{\tau-1}, (\hat{\theta}_\tau^i, \theta_\tau^{-i})) + \delta\mathbb{E}\left[\Phi_{\tau+1}^i(\boldsymbol{\theta}_t^{\tau-1}, (\hat{\theta}_\tau^i, \theta_\tau^{-i}), \tilde{\theta}_{\tau+1}) \middle| \chi_\tau(\hat{\theta}_\tau^i, \theta_\tau^{-i}), \theta_\tau\right] \tag{19}$$

for all $\tau > t$, $i \in \mathcal{I}$, $\boldsymbol{\theta}_t^{\tau-1} \in \times_{s=t}^\tau \Theta_s$, $\theta_\tau \in \Theta_\tau$ and $\hat{\theta}_t^i \neq \theta_t^i$

If we have the costless revelation condition in t, we can give each agent an arbitrary strong punishment for misreport at that time. If this condition is satisfied in the initial period, we can always make each agent' individual rationality fixed to be zero. Theorem 1 formally states this result.

Theorem 1 Suppose that $(\Theta_t, \mu_t, \chi_t)_{t=0}^\infty$ satisfies the costless revelation condition in $t = 0$, where χ is the efficient allocation rule. Suppose also that there exists a direct dynamic mechanism (χ, g), which is wp-EPIC and $(g_t^i)_{t=0}^\infty$ is uniformly bounded. Then, there exists a mechanism (χ, ψ) which is wp-EPIC, wp-EPIR and extracts the full surplus.

In order to prove Theorem 1, we firstly show Lemma 1, which allows us to focus on the individual rationality in the initial period.

Lemma 1 Suppose that there exists a bounded payment rule ϕ s.t. (χ, ϕ) is wp-EPIR, which yields the revenue R and satisfies wp-EPIR in period 0, i.e.,

$$V_0^i(\theta_0) + \Phi_0^i(\theta_0) \geq O_0^i(\theta_0) \tag{20}$$

holds for all $i \in \mathcal{I}$ and $\theta_0 \in \Theta_0$. Then, there exists a mechanism (χ, ψ) which is wp-EPIC, wp-EPIR and yields the revenue R.

Proof of Lemma 1 Define $d_t^i : \Theta_t \to \mathbb{R}$ by

$$d_t^i(\boldsymbol{\theta}_t) = \max_{\theta_{t+1} \in \Theta_{t+1}} \{O_{t+1}^i(\theta_{t+1}) - V_{t+1}^i(\theta_{t+1}) - \Phi_{t+1}^i(\boldsymbol{\theta}^t, \theta_{t+1})\} \tag{21}$$

for all $t \in \mathbb{N}$. By assumption, v_t^i, ϕ_t^i and O_t^i are uniformly bounded, so d_t^i is.

Define a new mechanism (χ, ψ) by

$$\psi_0^i(\theta_0) = \phi_0^i(\theta_0) - \delta d_0^i(\theta_0) \tag{22}$$

for $t = 0$ and

$$\psi_t^i(\boldsymbol{\theta}^t) = \phi_t^i(\boldsymbol{\theta}^t) - \delta d_t^i(\boldsymbol{\theta}^t) + d_{t-1}^i(\boldsymbol{\theta}^{t-1}) \tag{23}$$

Intuitively, $\delta d_t^i(\theta_t)$ is the deposit to the social planner and $d_{t-1}^i(\theta_{t-1})$ is withdrawal. Then, for any t and for any realizations of θ_t,

$$\begin{aligned} \Psi_t^i(\boldsymbol{\theta}^t) &= \lim_{T \to \infty} \mathbb{E}\left[\sum_{\tau=t}^{T} \delta^{\tau-t} \psi_\tau^i(\tilde{\boldsymbol{\theta}}^\tau) \middle| \boldsymbol{\theta}_t \right] \\ &= \lim_{T \to \infty} \mathbb{E}\left[-\delta^{T-t+1} d_T^i(\tilde{\boldsymbol{\theta}}^T) + d_{t-1}^i(\boldsymbol{\theta}^{t-1}) + \sum_{\tau=t}^{T} \delta^{\tau-t} \phi_\tau^i(\tilde{\boldsymbol{\theta}}^\tau) \middle| \boldsymbol{\theta}_t \right] \\ &= \Phi_t^i(\boldsymbol{\theta}^t) + d_t^i(\boldsymbol{\theta}^{t-1}) \end{aligned} \tag{24}$$

The last equality holds because uniform boundedness of d_t^i assures that $\delta^{T-t+1}\mathbb{E}[d_T^i(\tilde{\boldsymbol{\theta}}^T)] \to 0$ as $T \to \infty$. Then, agent i's EPV is

$$V_t^i(\theta_t) + \Psi_t^i(\boldsymbol{\theta}^t) = V_t^i(\theta_t) + \Phi_t^i(\boldsymbol{\theta}^t) + d_{t-1}^i(\boldsymbol{\theta}^{t-1}) \geq O^i(\theta_t). \tag{25}$$

Furthermore, since agent i cannot manipulate $\boldsymbol{\theta}^{t-1}$, we can treat $d_{t-1}^i(\boldsymbol{\theta}_{t-1})$ as a constant term. Therefore, the wp-EPIC in the original mechanism (χ, ϕ) is preserved in (χ, ψ).

Similarly, in the initial period, $\Psi_0^i(\theta_0) = \Phi_0^i(\theta_0)$ for all $\theta_0 \in \Theta_0$. Therefore, (χ, ψ) is wp-EPIC, wp-EPIR and yields the revenue R. ■

Proof of Theorem 1 By Lemma 1, it suffices to show that there exists a mechanism (χ, ψ) s.t. (i) (χ, ψ) gives each agent exactly the same EPV as the outside opportunity, i.e., $V_0^i(\theta_0) + \Psi_0^i(\theta_0) = O_0^i(\theta_0)$ for all $\theta_0 \in \Theta_0$ and (ii) (χ, ψ) is wp-EPIC, and (iii) ψ_t^i is uniformly bounded.

By assumption, there exists g s.t. (χ, g) is wp-EPIC and g_t^i is uniformly bounded. In addition, by the costless revelation condition in $t = 0$, it is guaranteed that there exists a part of payment rule ϕ which satisfies all the three conditions introduced in Definition 4. Using these g and ϕ, construct a new payment rule ψ as

$$\psi_0^i(\theta_0) = -v_0^i(\chi_0(\theta_0), \theta_0) - \delta\mathbb{E}\left[V_1^i(\tilde{\theta}_1) + G_1^i(\theta_0, \tilde{\theta}_1)\middle|\, \chi_0(\theta_0), \theta_0\right] + O_0^i(\theta_0) \tag{26}$$

for $t = 0$ and

$$\psi_t^i(\boldsymbol{\theta}^t) = g_t^i(\boldsymbol{\theta}^t) + \alpha^i(\theta_0^{-i})\phi_t^i(\boldsymbol{\theta}^t) \tag{27}$$

for $t \geq 1$, where $\alpha^i(\theta_0^- i) \in \mathbb{R}_{++}$ is a sufficiently large scalar (the detail is shown later).

Then, agent i's on-path EPV in $t = 0$ from the payment rule is

$$\Psi_0^i(\theta_0) = -V_0^i(\theta_0), \theta_0) + O_0^i(\theta_0) \tag{28}$$

for all θ_0, so the first condition is satisfied.

Next, we want to show that (χ, ψ) is wp-EPIC. When $t \geq 1$, wp-EPIC is obviously hold since (χ, g) satisfies wp-EPIC and ϕ does not hurt the incentive structure. If agent i makes one-shot deviation to $\hat{\theta}_0^i$ in $t = 0$, his EPV is

$$\begin{aligned}
&v_0^i(\chi_0(\hat{\theta}_0^i, \theta_0^{-i}), \theta_0) + \delta\mathbb{E}\left[V_1^i(\tilde{\theta}_1) + G_1^i((\hat{\theta}_0^i, \theta_0^{-i}), \tilde{\theta}_1)\middle|\, \chi_0(\hat{\theta}_0^i, \theta_0^{-i}), \theta_0\right] \\
&- v_0^i(\chi_0(\hat{\theta}_0^i, \theta_0^{-i}), (\hat{\theta}_0^i, \theta_0^{-i})) - \delta\mathbb{E}\left[V_1^i(\tilde{\theta}_1) + G_1^i((\hat{\theta}_0^i, \theta_0^{-i}), \tilde{\theta}_1)\middle|\, \chi_0(\hat{\theta}_0^i, \theta_0^{-i}), (\hat{\theta}_0^i, \theta_0^{-i})\right] \\
&+ O_0^i(\hat{\theta}_0^i, \theta_0^{-i}) + \delta\alpha^i(\theta_0^{-i})\mathbb{E}\left[\Phi_1^i((\hat{\theta}_0^i, \theta_0^{-i}), \tilde{\theta}_1)\middle|\, \chi(\hat{\theta}_0^i, \theta_0^{-i}), \theta_0\right]
\end{aligned} \tag{29}$$

Let

$$\alpha^i(\theta_0^{-i}) = \max_{\substack{\theta_0^i, \hat{\theta}_0^i \in \Theta_0^i \\ \theta_0^i \neq \hat{\theta}_0^i}} \frac{\left\{\begin{aligned}
&- v_0^i(\chi_0(\hat{\theta}_0^i, \theta_0^{-i}), \theta_0) + v_0^i(\chi_0(\hat{\theta}_0^i, \theta_0^{-i}), (\hat{\theta}_0^i, \theta_0^{-i})) \\
&- \delta\mathbb{E}\left[V_1^i(\tilde{\theta}_1) + G_1^i((\hat{\theta}_0^i, \theta_0^{-i}), \tilde{\theta}_1)\middle|\, \chi_0(\hat{\theta}_0^i, \theta_0^{-i}), \theta_0\right] \\
&+ \delta\mathbb{E}\left[V_1^i(\tilde{\theta}_1) + G_1^i((\hat{\theta}_0^i, \theta_0^{-i}), \tilde{\theta}_1)\middle|\, \chi_0(\hat{\theta}_0^i, \theta_0^{-i}), (\hat{\theta}_0^i, \theta_0^{-i})\right] \\
&- O_0^i(\hat{\theta}_0^i, \theta_0^{-i}) + O_0^i(\theta_0)
\end{aligned}\right\}}{\delta\left|\mathbb{E}\left[\Phi_1^i((\hat{\theta}_0^i, \theta_0^{-i}), \tilde{\theta}_1)\middle|\, \chi_0(\hat{\theta}_0^i, \theta_0^{-i}), \theta_0\right]\right|}. \tag{30}$$

Then his EPV from any deviation becomes smaller than or equal to $O_0^i(\theta_0)$, which is his on-path EPV.

Finally, since all of v_t^i, g_t^i, ϕ_t^i, and O_t^i's are uniformly bounded, ψ_t^i are also bounded uniformly. Applying Lemma 1, we obtain a mechanism which is wp-EPIC, wp-EPIR and extracts the full surplus. ■

Although individual rationality in later periods can be solved by "borrowing and lending agreement," if the amount of the deposit increases faster than inverse of the discount rate, the EPV of transaction is not zero. Uniform boundedness of the valuation functions and payments are one of the most tractable sufficient conditions which guarantee no Ponzi game condition here.

Many preceding papers studied the dynamic counterparts of the Groves mechanisms. Above all, the *Team mechanism* suggested by Athey and Segal (2013) is the most suitable for the "original mechanism" (χ, g) in Theorem 1.

Theorem 2 (the Team mechanism: Athey and Segal (2013)) Suppose that we have *private values*, i.e., for any $i \in \mathcal{I}$, $t \in \mathbb{N}$, v_t^i is irrelevant to $\theta_t^j \in \Theta_t^j$ for all $j \in \mathcal{I} \backslash \{i\}$. Suppose also that χ is the efficient allocation rule. Then, (χ, g) where $g_t^i(\theta_t) = \sum_{j \in \mathcal{I} \backslash \{i\}} v_t^j(\chi(\theta_t), \theta_t)$ is wp-EPIC.

Proof See Athey and Segal (2013).

Athey and Segal (2013) named this (χ, g) the Team mechanism. Since we assume that v_t^i's are uniformly bounded, so g_t^i's are. Combining Theorem 1 and 2, we obtain Corollary 1, since this mechanism is wp-EPIC regardless of the shape of μ.

Corollary 1 Suppose that $(\Theta_t, \mu_t, \chi_t)_{t=0}^{\infty}$ satisfies the costless revelation condition in $t = 0$, where χ is the efficient allocation rule. Suppose also that we have private values. Then, there exists a mechanism which is wp-EPIC, wp-EPIR and extracts the full surplus.

5 Full Rankness and Costless Revelation

In section 4, we have shown that the costless revelation condition in the initial period is the key for possibility of full surplus extraction. Here a question naturally arises. When is the costless revelation condition satisfied? As is shown in the short period examples, we can reveal each agent's private information without leaving information rent when agents' states are intertemporally correlated.

Similar to static environments, in order to apply the Crémer-McLean scheme, a dynamic version of *full rank condition* is needed. For notational simplicity, here we interpret each $\mu_t(x_{t-1}, \theta_{t-1}) \in \Delta(\Theta_t)$ ($\mu_t^{-i}(x_{t-1}, \theta_{t-1}) \in \Delta(\Theta_t^{-i})$) as a $|\Theta_t|$-length ($|\Theta_t^{-i}|$-length) probability vector.

Definition 5 (Intertemporal Full Rankness) $(\Theta_\tau, \mu_\tau)_{\tau=0}^{\infty}$ satisfies the *full rank condition in t* if one of the following conditions are satisfied.

(i) For any i, $x_t \in X_t$, and $\theta_t^{-i} \in \Theta_t^{-i}$, $\{\mu_{t+1}^{-i}(x_t, (\theta_t^i, \theta_t^{-i}))\}_{\theta_t^i \in \Theta_t^i}$ are linearly independent.

(ii) For any i, $x_t \in X_t$, and θ_t^{-i}, $\{\mu_{t+1}(x_t, (\theta_t^i, \theta_t^{-i}))\}_{\theta_t^i \in \Theta_t^i}$ are linearly independent and $(\Theta_\tau, \mu_\tau)_{\tau=0}^{\infty}$ satisfies the full rank condition in $t+1$.

The first condition requires that given any θ_t^{-i}, θ_t^i has some information about the other agents' future states θ_{t+1}^{-i}. Since this agent cannot manipulate the report of θ_{t+1}^{-i}, the first condition itself is sufficient for constructing a Crémer-McLean lottery. This is the mechanism designer's first chance, and it generically holds as long as $|\Theta_{t+1}^{-i}| \geq |\Theta_t^i|$.

Even when the first condition fails, we still have a chance. The second condition is given any θ_t^{-i}, θ_t^i has full rank correlation with something which will realize tomorrow, and the mechanism designer can surely *reveal the true realization* of it. Although the former part of the second condition is weaker than the first one, it is not sufficient by itself to produce a Crémer-McLean lottery because the mechanism designer might not detect the agent's *contingent deviation*. However, if the agent foresights that he does not have any profitable deviation in the next period, there is no essential difference between θ_{t+1}^i and θ_{t+1}^{-i} because the true realizations of both are surely unveiled tomorrow. Therefore, given the full rank condition in the next period, the mechanism designer has a rather rich information structure.

It is obvious that linear independence of θ_t^{-i}, $\{\mu_{t+1}(x_t, (\theta_t^i, \theta_t^{-i}))\}_{\theta_t^i \in \Theta_t^i}$ is always weaker than that of $\theta_t^{-i} \in \Theta_t^{-i}$, $\{\mu_{t+1}^{-i}(x_t, (\theta_t^i, \theta_t^{-i}))\}_{\theta_t^i \in \Theta_t^i}$. Note that the second condition might be satisfied even when we have *independent types* in the dynamic sense in $t+1$, i.e., the transition probability can be written in the form that $\mu_{t+1}(x_t, \theta_t) = \mu_{t+1}^0(x_t, \theta_t^0) \cdot \prod_{i \in \mathcal{I}} \mu_{t+1}^i(x_t, \theta_t^0, \theta_t^i)$ [4], while there is no hope for the first one. In reality, this generalization is much more than a mathematical, dimensional extension. Without controversy, it is natural to expect that the most promising clue to predict θ_{t+1}^i is θ_t^i and sometimes θ_t^i is useless for predicting θ_{t+1}^{-i}. A typical example is persistent types, shown as the three-period example.

The intertemporal full rank condition is sufficient for costless revelation. Thereom 3 states this result.

Theorem 3 Suppose that $(\Theta_\tau, \mu_\tau)_{\tau=0}^\infty$ satisfies the full rank condition in t. Then, for any χ, $(\Theta_\tau, \mu_\tau, \chi_\tau)_{\tau=0}^\infty$ satisfies the costless revelation condition in t.

Proof of Theorem 3 If (i) of the full rank condition is satisfied, $\mu_{t+1}^{-i}(\chi_t(\theta_t), \theta_t)$ is not in the convex hull of $\{\mu_{t+1}^{-i}(\chi_t(\theta_t), (\hat{\theta}_t^i, \theta_t^{-i}))\}_{\hat{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}$. Applying the separating hyperplane theorem, there exists a $|\Theta_{t+1}^{-i}|$-length vector $b_{t+1}^i(\theta_t)$ s.t.

$$b_{t+1}^i(\theta_t) \cdot \mu_{t+1}^{-i}(\chi_t(\theta_t), \theta_t) > b_{t+1}^i(\theta_t) \cdot \mu_{t+1}^{-i}(\chi_t(\theta_t), (\hat{\theta}_t^i, \theta_t^{-i})), \tag{31}$$

equivalently,

$$\mathbb{E}\left[b_{t+1}^i(\theta_t)(\tilde{\theta}_{t+1}^{-i}) \middle| \chi_t(\theta_t), \theta_t \right] > \mathbb{E}\left[b_{t+1}^i(\theta_t)(\tilde{\theta}_{t+1}^{-i}) \middle| \chi_t(\theta_t), (\hat{\theta}_t^i, \theta_t^{-i}) \right] \tag{32}$$

holds for all $\hat{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}$, and $||b_{t+1}^i(\theta_t)|| = 1$.

Construct ϕ in a following way. $\phi_{t+1}^i : \Theta_t \times \Theta_{t+1}^{-i} \to \mathbb{R}$ is defined as

$$\phi_{t+1}^i(\theta_t, \theta_{t+1}^{-i}) = b_{t+1}^i(\theta_t)(\theta_{t+1}^{-i}) - \mathbb{E}\left[b_{t+1}^i(\theta_t)(\tilde{\theta}_{t+1}^{-i}) \middle| \chi(\theta_t), \theta_t \right], \tag{33}$$

and $\phi_\tau^i(\boldsymbol{\theta}_t^\tau) = 0$ for all $\tau > t+1$ and $\boldsymbol{\theta}_t^\tau$. Then,

$$\begin{aligned} &\mathbb{E}\left[\Phi_t^i(\theta_t, \tilde{\theta}_t^{-i}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t \right] \\ &= \mathbb{E}\left[b_{t+1}^i(\tilde{\theta}_{t+1}^{-i}) \middle| \chi_t(\theta_t), \theta_t \right] - \mathbb{E}\left[b_{t+1}^i(\tilde{\theta}_{t+1}^{-i}) \middle| \chi_t(\theta_t), \theta_t \right] = 0 \end{aligned} \tag{34}$$

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[4] If this holds for all t, it becomes the definition of *independent types* in Athey and Segal (2013), and a basic assumption in Bergemann and Välimäki (2010) and Cavallo et al. (2010). Note that this condition is stronger than conditional independence, since the distribution of θ_{t+1}^i cannot depend on θ_t^j for all $j \in \mathcal{I} \setminus \{i\}$.

and

$$\begin{aligned}
&\mathbb{E}\left[\Phi_t^i(\theta_t,\tilde{\theta}_t^{-i})\middle|\,\chi(\hat{\theta}_t^i,\theta_t^{-i}),\theta_t\right]\\
&=\mathbb{E}\left[b_{t+1}^i(\tilde{\theta}_{t+1}^{-i})\middle|\,\chi_t(\hat{\theta}_t^i,\theta_t^{-i}),\theta_t\right]-\mathbb{E}\left[b_{t+1}^i(\tilde{\theta}_{t+1}^{-i})\middle|\,\chi_t(\hat{\theta}_t^i,\theta_t^{-i}),(\hat{\theta}_t^i,\theta_t^{-i})\right]<0
\end{aligned}\tag{35}$$

holds. In addition, since agent i's report after t does not affect ϕ_τ^i's, so equation (19) trivially holds.

Suppose that the full rank condition is satisfied with the second statement. First, we will prove the following lemma.

Lemma 2 Suppose that for any i, $x_t \in X_t$, and θ_t^{-i}, $\{\mu_{t+1}(x_t,(\theta_t^i,\theta_t^{-i}))\}_{\theta_t^i\in\Theta_t^i}$ are linearly independent and for any χ, $(\Theta_\tau,\mu_\tau,\chi_\tau)_{\tau=0}^\infty$ satisfies the costless revelation condition in $t+1$. Then, costless revelation is also possible in t.

Proof of Lemma 2 By assumption, for any i, $\theta_t \in \Theta_t$, $\mu_{t+1}^{-i}(\chi_t(\theta_t),\theta_t)$ is not in the convex hull of $\{\mu_{t+1}(\chi_t(\theta_t),(\hat{\theta}_t^i,\theta_t^{-i}))\}_{\hat{\theta}_t^i\in\Theta_t^i\backslash\{\theta_t^i\}}$. Applying the separating hyperplane theorem, there exists a $|\Theta_{t+1}|$-length vector $b_{t+1}^i(\theta_t)$ s.t.

$$\mathbb{E}\left[b_{t+1}^i(\theta_t)(\tilde{\theta}_{t+1})\middle|\,\chi_t(\theta_t),\theta_t\right]>\mathbb{E}\left[b_{t+1}^i(\theta_t)(\tilde{\theta}_{t+1})\middle|\,\chi_t(\theta_t),(\hat{\theta}_t^i,\theta_t^{-i})\right]\tag{36}$$

holds for all $\hat{\theta}_t^i \in \Theta_t^i\backslash\{\theta_t^i\}$, and $||b_{t+1}(\theta_t)||=1$. Since costless revelation is possible in $t+1$ by assumption, there exists $((w_\tau^i)_{i\in\mathcal{I}})_{\tau=t+2}^\infty$ which satisfies all the three conditions of Definition 4. Construct ϕ as follows.

$$\phi_{t+1}^i(\theta_t,\theta_{t+1})=b_{t+1}^i(\theta_t)(\theta_{t+1})-\mathbb{E}\left[b_{t+1}^i(\theta_t)(\tilde{\theta}_{t+1})\middle|\,\chi_t(\theta_t),\theta_t\right],\tag{37}$$

and

$$\phi_\tau^i(\boldsymbol{\theta}_t^\tau)=\alpha^i(\theta_t,\theta_{t+1}^{-i})w_\tau^i(\boldsymbol{\theta}_{t+1}^\tau)\tag{38}$$

for all $\tau > t+1$. $\alpha^i(\theta_t,\theta_{t+1}) \in \mathbb{R}_{++}$ is a sufficiently large scalar (construction of α^i will be stated in detail later).

Then, for all $\tau > t+1$, truthtelling is a best response regardless of the history, by construction of w_τ^i. Consider agent i's problem in period $t+1$. Given any yesterday's reported state profile θ_t, and today's realization θ_{t+1}, his EPV from truthful report is

$$\Phi_{t+1}^i(\theta_t,\theta_{t+1})=b_{t+1}^i(\theta_t)(\theta_{t+1})-\mathbb{E}\left[b_{t+1}^i(\theta_t)(\tilde{\theta}_{t+1})\middle|\,\chi_t(\theta_t),\theta_t\right].\tag{39}$$

If he makes a one-shot deviation to $\hat{\theta}_{t+1}^i \neq \theta_{t+1}^i$, his EPV from ϕ is

$$\begin{aligned}
&\phi_{t+1}^i(\theta_t,(\hat{\theta}_{t+1}^i,\theta_{t+1}^{-i}))+\delta\mathbb{E}\left[\Phi_{t+2}^i(\theta_t,(\hat{\theta}_{t+1}^i,\theta_{t+1}^{-i}),\tilde{\theta}_{t+2})\middle|\,\chi_{t+1}(\hat{\theta}_{t+1}^i,\theta_{t+1}^{-i}),\theta_{t+1}\right]\\
&=b_{t+1}^i(\theta_t)(\hat{\theta}_{t+1}^i,\theta_{t+1}^{-i})-\mathbb{E}\left[b_{t+1}^i(\theta_t)(\tilde{\theta}_{t+1})\middle|\,\chi_t(\theta_t),\theta_t\right]\\
&+\delta\alpha^i(\theta_t,\theta_{t+1}^{-i})\mathbb{E}\left[W_{t+2}^i((\hat{\theta}_{t+1}^i,\theta_{t+1}^{-i}),\tilde{\theta}_{t+2})\middle|\,\chi_{t+1}(\hat{\theta}_{t+1}^i,\theta_{t+1}^{-i}),\theta_{t+1}\right]
\end{aligned}\tag{40}$$

By construction of w, $\mathbb{E}\left[W^i_{t+2}(\theta_t, (\hat{\theta}^i_{t+1}, \theta^{-i}_{t+1}), \tilde{\theta}_{t+2}) \middle| \chi_{t+1}(\hat{\theta}^i_{t+1}, \theta^{-i}_{t+1}), \theta_{t+1}\right] < 0$. Then, defining $\alpha^i(\theta_t, \theta^{-i}_{t+1})$ as

$$\begin{aligned} &\alpha^i(\theta_t, \theta^{-i}_{t+1}) \\ = &\max_{\substack{\theta^i_{t+1}, \hat{\theta}^i_{t+1} \in \Theta^i_{t+1} \\ \theta^i_{t+1} \neq \hat{\theta}^i_{t+1}}} \frac{b^i_{t+1}(\theta_t)(\theta_{t+1}) - b^i_{t+1}(\theta_t)(\hat{\theta}^i_{t+1}, \theta^{-i}_{t+1})}{\delta \left| \mathbb{E}\left[W^i_{t+2}((\hat{\theta}^i_{t+1}, \theta^{-i}_{t+1}), \tilde{\theta}_{t+2}) \middle| \chi_{t+1}(\hat{\theta}^i_{t+1}, \theta^{-i}_{t+1}), \theta_{t+1}\right]\right|}, \end{aligned} \tag{41}$$

each agent does not have any profitable deviation in $t+1$. The incentive condition in period t can be verified in a similar way to the first part of the proof of Theorem 3. ■

Proof of Theorem 3 (continued) Given Lemma 2, the remaining part of Theorem 3 is immediate. If the full rank condition is satisfied with the second statement in t, there exists some $t^* > t$ when it is satisfied with the first one. Then, by the first part of the proof of Theorem 3, costless revelation is possible in t^*. Applying Lemma 2 iteratively, it is shown that costless revelation is also possible in t. ■

6 Implementation by Costless Revelation

As we argued in Section 4, in order to extract the whole surplus, we have to (i) implement the efficient allocation, and (ii) reveal the initial state profile without leaving information rent. Theorem 1 and 3 imply that the intertemporal full rank condition in the first period guarantees the second part of these requirements. In fact, what is better, the full rank condition is also useful to meet the first requirement.

Theorem 4 (Implementation) Suppose that $(\Theta_\tau, \mu_\tau, \chi_\tau)_{\tau=0}^\infty$ satisfies the costless revelation condition for all $t \in \mathbb{N}$. Then, there exists a payment rule ψ which makes (χ, ψ) wp-EPIC.

Proof of Theorem 4 Since the costless revelation condition holds for all $t \in \mathbb{N}$, there exist $(\phi^i_{t,\tau})_{\tau=t+1}^\infty$ which satisfies all the three condition in Definition 4, for each t. Define the new payment rule ψ as $\psi^i_0(\theta_0) = -v^i_0(\chi_0(\theta_0), \theta_0)$ and

$$\psi^i_t(\boldsymbol{\theta}^t) = -v^i_t(\chi_t(\theta_t), \theta_t) + \sum_{s=0}^{t-1} \alpha^i_s(\theta^{-i}_s) \phi^i_{s,t}(\boldsymbol{\theta}^t_s). \tag{42}$$

for each $t \geq 1$.

Then, for all t and $\boldsymbol{\theta}^{t-1}$, agent i's EPV from truthful reports is

$$V^i_t(\theta_t) + \Psi^i_t(\boldsymbol{\theta}^t) = \sum_{s=0}^{t-1} \alpha^i_s(\theta^{-i}_s) \Phi^i_{s,t}(\boldsymbol{\theta}^t_s). \tag{43}$$

If he makes a one-shot deviation to $\hat{\theta}_t^i \neq \theta_t^i$,

$$\begin{aligned}
& v_t^i(\chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t) + \psi_t^i(\boldsymbol{\theta}^{t-1}, (\hat{\theta}_t^i, \theta_t^{-i})) \\
+ \quad & \delta \mathbb{E}\left[V_{t+1}^i(\tilde{\theta}_{t+1}) + \Psi_t^i(\boldsymbol{\theta}^{t-1}, (\hat{\theta}_t^i, \theta_t^{-i}), \tilde{\theta}_{t+1}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t \right] \\
= \quad & v_t^i(\chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t) - v_t^i(\chi_t(\hat{\theta}_t^i, \theta_t^{-i}), (\hat{\theta}_t^i, \theta_t^{-i})) && (44) \\
+ \quad & \sum_{s=0}^{t-1} \alpha_s^i(\theta_s) \left\{ \phi_{s,t}^i(\boldsymbol{\theta}_s^{t-1}, (\hat{\theta}_t^i, \theta_t^{-i})) + \delta \mathbb{E}\left[\Phi_{s,t+1}^i(\boldsymbol{\theta}_s^{t-1}, (\hat{\theta}_t^i, \theta_t^{-i}), \tilde{\theta}_{t+1}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t \right] \right\} && (45) \\
+ \quad & \delta \alpha_t^i(\theta_t^{-i}) \mathbb{E}\left[\Phi_{t,t+1}((\hat{\theta}_t^i, \theta_t^{-i}), \tilde{\theta}_{t+1}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t \right] && (46)
\end{aligned}$$

By construction of ϕ_s's, (43) is always smaller than or equal to (45). Therefore, defining

$$\alpha_t^i(\theta_t^{-i}) = \max_{\substack{\theta_t^i, \hat{\theta}_t^i \in \Theta_t^i \\ \theta_t^i \neq \hat{\theta}_t^i}} \frac{-v_t^i(\chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t) + v_t^i(\chi_t(\hat{\theta}_t^i, \theta_t^{-i}), (\hat{\theta}_t^i, \theta_t^{-i}))}{\delta \left| \mathbb{E}\left[\Phi_{t,t+1}^i((\hat{\theta}_t^i, \theta_t^{-i}), \tilde{\theta}_{t+1}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t \right] \right|}, \tag{47}$$

wp-EPIC holds in each t. ■

When we are considering a finite-horizon problem, it also implies possibility of full surplus extraction immediately. However, unfortunately it is not always the case for an infinite-horizon problem. Our strategy for keeping wp-EPIR in later periods is the deposit scheme, and if the amount of deposit increases without bound and faster than discount rate, it hurts the incentive structure. In order to avoid this, we need a slightly stronger rank condition.

Definition 6 (Uniform Intertemporal Full Rankness) Let ρ be the Euclidean distance. $(\Theta_t, \mu_t)_{t=0}^{\infty}$ satisfies the *uniform full rank condition* if there exists $k > 0$ s.t.

(i) there exists $T \in \mathbb{N}$ s.t. for all $t \in \mathbb{N}$, there exists $t \leq t^* \leq T$ s.t. for all $x_{t^*} \in X_{t^*}$ and $\theta_{t^*} \in \Theta_{t^*}$,

$$\rho\left(\mu_{t^*+1}^{-i}(x_{t^*}, \theta_{t^*}), \mathrm{conv}\{\mu_{t^*+1}^{-i}(x_{t^*}, (\hat{\theta}_{t^*}^i, \theta_{t^*}^{-i}))\}_{\hat{\theta}_{t^*}^i \in \Theta_{t^*}^i \setminus \{\theta_{t^*}^i\}}\right) > k \tag{48}$$

(ii) for all $t \in \mathbb{N}$, $x_t \in X_t$, $\theta_t \in \Theta_t$

$$\rho\left(\mu_{t+1}(x_t, \theta_t), \mathrm{conv}\{\mu_{t+1}(x_t, (\hat{\theta}_t^i, \theta_t^{-i}))\}_{\hat{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}\right) > k \tag{49}$$

Theorem 5 Suppose that we have the uniform full rank condition. Then full surplus extraction is guaranteed, i.e., for any $((v_t^i)_{i\in\mathcal{I}})_{t=0}^{\infty}$ and $((O_t^i)_{i\in\mathcal{I}})_{t=0}^{\infty}$, there exists a direct mechanism which is wp-EPIC, wp-EPIR and extract the full surplus.

Proof See Appendix.

It is well known that as correlation becomes weaker, the maximal payment of a Crémer-McLean lottery gets larger and larger. We can interpret this situation as an *asymptotically* short rank environment. Even when $\{\mu^{-i}_{t^*+1}(x_{t^*}, (\theta^i_{t^*}, \theta^{-i}_{t^*}))\}_{\theta^i_{t^*} \in \Theta^i_{t^*}}$ are linearly independent, these probability vectors get closer and closer, and coincide in the limit, the deposit scheme may skew the incentive structure. In order to avoid this problem, one of the most tractable and straightforward assumption is to set a minimal difference between a probability vector and the convex hull of the others. Note that our uniform full rank condition is equivalent to full rank condition for all t for finite-horizon problems.

7 Conclusion and Discussion

In this paper, we have studied dynamic mechanisms which extract the full surplus. To analyze it, it is effective to divide it into two different problems; (i) implement the efficient allocation rule, and (ii) extract whole the surplus and incentive subsidies. Making use of intertemporal correlation between signals, we can construct a Crémer-McLean like payment rule which reveals each agent's private information in the initial period. Moreover, it immediately solves the second requirement for full surplus extraction. If the implementation of the efficient allocation is guaranteed by some other scheme, like dynamic Groves mechanisms, costless revelation (derived from full rank intertemporal correlation) in the initial period is sufficient for full surplus extraction. We can also implement the efficient allocation by utilizing intertemporal correlation. Indeed, if costless revelation is possible in each period, implementability of the efficient allocation is assured. Combining these results, we can articulate a sufficient condition for full surplus extraction in dynamic environments.

Taken together, the precondition for full surplus extraction in a dynamic environment is weaker than the one in a static environment. Even when the state distribution is a conditionally independent, we still have a chance. Furthermore, even if we have "independent types" in the dynamic sense for most of periods, we may realize full surplus extraction by making use of state correlation in some exceptional periods.

This result gives us a very clear and important message for the field of mechanism design. Sometimes we can obtain a desirable outcome, which was thought to be impossible to produce, by utilizing the environment's dynamic nature. In order to make the most of dynamic nature, we may have to design a complicated mechanism beforehand. As a matter of fact, our result suggests that it is not sufficient to examine payment rules which are associated only with reports made yesterday and today.

Our mechanisms utilize all the existing intertemporal correlation. Therefore, in our conjecture, the necessary condition for full surplus extraction may be quite close to our full rank condition. However, it is much more complicated problem than the one in a static environment, since we have to investigate a very rich class of payment rules, which can depend on all the past reports. While it is very interesting and important problem, we leave this problem for future research.

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Appendix

Proof of Theorem 5

Step 1 There exists $T < \infty$ and $k > 0$ s.t. for all $t \in \mathbb{N}$ there exists t^* s.t. $t < t^* < T$ and there exists $(\phi_{t^*,\tau})_{\tau=t^*+1}^{\infty}$ s.t.

(i) $(\phi_{t^*,\tau})_{\tau=t^*+1}^{\infty}$ is an appropriate costless revelation payment rule in t^*, i.e., it satisfies all the three conditions in Definition 4,

(ii) $(\phi_{t^*,\tau})_{\tau=t^*+1}^{\infty}$ has a uniform upper bound $\bar{\phi}_0 < \infty$,

(iii) $\phi^i_{t^*,\tau}(\boldsymbol{\theta}^{\tau}_{t^*+1}) = 0$ for all $\tau \geq t^* + 2$,

(iv) there exists $\epsilon > 0$ s.t. $\left|\mathbb{E}\left[\Phi^i_{t^*,t^*+1}((\hat{\theta}^i_{t^*}, \theta^{-i}_{t^*}), \tilde{\theta}_{t^*+1}) \,\middle|\, \chi_{t^*}(\hat{\theta}^i_{t^*}, \theta^{-i}_{t^*}), \theta_{t^*}\right]\right| > \epsilon$ for all $\theta^i_{t^*} \neq \hat{\theta}^i_{t^*}$ and $\theta^{-i}_{t^*} \in \Theta^{-i}_{t^*}$.

Proof of Step 1 By assumption, there exists $T \in \mathbb{N}$ s.t. for all $t \in \mathbb{N}$, there exists $t \leq t^* \leq T$ s.t. for all $x_{t^*} \in X_{t^*}$ and $\theta_{t^*} \in \Theta_{t^*}$,

$$\rho\left(\mu^{-i}_{t^*+1}(x_{t^*}, \theta_{t^*}), \mathrm{conv}\{\mu^{-i}_{t^*+1}(x_{t^*}, (\hat{\theta}^i_{t^*}, \theta^{-i}_{t^*}))\}_{\hat{\theta}^i_{t^*} \in \Theta^i_{t^*} \setminus \{\theta^i_{t^*}\}}\right) > k \tag{50}$$

Fix any such a t^* arbitrarily. We will construct a particular $b^i_{t^*+1} : \Theta_{t^*} \to \Delta(\Theta^{-i}_{t^*+1})$ which separates $\mu^{-i}_{t^*+1}(x_{t^*}, \theta_{t^*})$ and $\mathrm{conv}\{\mu^{-i}_{t^*+1}(x_{t^*}, (\hat{\theta}^i_{t^*}, \theta^{-i}_{t^*}))\}_{\hat{\theta}^i_{t^*} \in \Theta^i_{t^*} \setminus \{\theta^i_{t^*}\}}$ together with an appropriate constant, and $||b^i_{t^*+1}(\theta_{t^*})|| = 1$ for all $\theta_{t^*} \in \Theta_{t^*}$.

Take any $\theta_{t^*} \in \Theta_{t^*}$. Since $\mathrm{conv}\{\mu^{-i}_{t^*+1}(\chi_{t^*}(\theta_{t^*}), (\hat{\theta}^i_{t^*}, \theta^{-i}_{t^*}))\}_{\hat{\theta}^i_{t^*} \in \Theta^i_{t^*} \setminus \{\theta^i_{t^*}\}}$ is compact, there exists a point $\lambda^i_{t^*}(\chi_{t^*}(\theta_{t^*}), \theta_{t^*}) \in \mathrm{conv}\{\mu^{-i}_{t^*+1}(\chi_{t^*}(\theta_{t^*}), (\hat{\theta}^i_{t^*}, \theta^{-i}_{t^*}))\}_{\hat{\theta}^i_{t^*} \in \Theta^i_{t^*} \setminus \{\theta^i_{t^*}\}}$ which is closest to $\mu^{-i}_{t^*+1}(x_{t^*}, \theta_{t^*})$. Define

$$b^i_{t^*+1}(\theta_{t^*}) = \frac{\mu^{-i}_{t^*+1}(\chi_{t^*}(\theta_{t^*}), \theta_{t^*}) - \lambda^i_{t^*}(\chi_{t^*}(\theta_{t^*}), \theta_{t^*})}{||\mu^{-i}_{t^*+1}(\chi_{t^*}(\theta_{t^*}), \theta_{t^*}) - \lambda^i_{t^*}(\chi_{t^*}(\theta_{t^*}), \theta_{t^*})||} \tag{51}$$

Then, clearly $||b^i_{t^*+1}(\theta_{t^*})|| = 1$. Furthermore,

$$\begin{aligned} & b^i_{t^*+1}(\theta_{t^*}) \cdot \left[\mu^{-i}_{t^*+1}(\chi_{t^*}(\theta_{t^*}), \theta_{t^*}) - \lambda^i_{t^*}(\chi_{t^*}(\theta_{t^*}), \theta_{t^*})\right] \\ &= \left|\left|\mu^{-i}_{t^*+1}(\chi_{t^*}(\theta_{t^*}), \theta_{t^*}) - \lambda^i_{t^*}(\chi_{t^*}(\theta_{t^*}), \theta_{t^*})\right|\right| \\ &> k \end{aligned} \tag{52}$$

and

$$b^i_{t^*+1}(\theta_{t^*}) \cdot \left[\lambda^i_{t^*}(\chi_{t^*}(\theta_{t^*}), \theta_{t^*}) - \mu^{-i}_{t^*+1}(\chi_{t^*}(\theta_{t^*}), (\hat{\theta}^i_{t^*}, \theta^{-i}_{t^*}))\right] \leq 0 \tag{53}$$

holds, since $b^i_{t^*+1}(\theta_{t^*})$ and $\lambda^i_{t^*}(\chi_{t^*}(\theta_{t^*}),\theta_{t^*})-\mu^{-i}_{t^*+1}(\chi_{t^*}(\theta_{t^*}),(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*}))$ cannot make an acute angle. Combining (52) and (53), we obtain

$$b^i_{t^*+1}(\theta_{t^*})\cdot\left[\mu^{-i}_{t^*+1}(\chi_{t^*}(\theta_{t^*}),\theta_{t^*})-\mu^{-i}_{t^*+1}(\chi_{t^*}(\theta_{t^*}),(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*}))\right]>k \tag{54}$$

for all $\hat{\theta}^i_{t^*}\neq\theta^i_{t^*}$.

As is shown in the first part of the proof of Theorem 3, defining $(\phi_{t^*,\tau})^{\infty}_{\tau=t^*+1}$ as

$$\phi^i_{t^*,t^*+1}(\theta_{t^*},\theta^{-i}_{t^*+1})=b^i_{t^*+1}(\theta_{t^*})(\theta^{-i}_{t^*+1})-\mathbb{E}\left[b^i_{t^*+1}(\theta_{t^*})(\tilde{\theta}^{-i}_{t^*+1})\middle|\,\chi_{t^*}(\theta_{t^*}),\theta_{t^*}\right] \tag{55}$$

and $\phi_{t^*,\tau}(\boldsymbol{\theta}^{\tau}_{t^*})=0$ for all $\tau>t^*+2$, this $(\phi_{t^*,\tau})^{\infty}_{\tau=t^*+1}$ reveals the states costlessly, has an upper bound $\bar{\phi}_0=2$, and irrelevant to the report after t^*+2. In addition,

$$\begin{aligned}
&\left|\mathbb{E}\left[\Phi^i_{t^*,t^*+1}((\hat{\theta}^i_{t^*},\theta^{-i}_{t^*}),\tilde{\theta}_{t^*+1})\middle|\,\chi_{t^*}(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*}),\theta_{t^*}\right]\right|\\
=\ &\delta\left|\mathbb{E}\left[b^i_{t^*+1}(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*})(\tilde{\theta}^{-i}_{t^*+1})\middle|\,\chi_{t^*}(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*}),\theta_{t^*}\right]\right.\\
&\qquad\qquad\left.-\ \mathbb{E}\left[b^i_{t^*+1}(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*})(\tilde{\theta}^{-i}_{t^*+1})\middle|\,\chi_{t^*}(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*}),(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*})\right]\right|\\
=\ &\delta\left|b^i_{t^*+1}(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*})\cdot\left[\mu^{-i}_{t^*+1}(\chi_{t^*}(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*}),\theta_{t^*})-\mu^{-i}_{t^*+1}(\chi_{t^*}(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*}),(\hat{\theta}^i_{t^*},\theta^{-i}_{t^*})\right]\right|\\
>\ &\delta k=\epsilon.\ \blacksquare
\end{aligned} \tag{56}$$

Step 2 Let $0\leq t^*(t)\leq T$ be the closest future start point for t. There exists $(\bar{\phi}_0,\bar{\phi}_1,\bar{\phi}_2,...)$ and $\epsilon>0$ s.t. for any $t\in\mathbb{N}$ s.t. $n=t^*(t)-t$, there exists $(\phi_{t,\tau})^{\infty}_{\tau=t+1}$ s.t.

(i) $(\phi_{t,\tau})^{\infty}_{\tau=t+1}$ is an appropriate costless revelation payment rule in t, i.e., it satisfies all the three conditions in Definition 4,

(ii) $\bar{\phi}_n$ is a uniform upper bound of $(\phi_{t,\tau})^{\infty}_{\tau=t+1}$,

(iii) $\phi^i_{t,\tau}(\boldsymbol{\theta}^{\tau}_t)=0$ for all $\tau\geq t+n+1$,

(iv) $\left|\mathbb{E}\left[\Phi^i_{t,t+1}((\hat{\theta}^i_t,\theta^{-i}_t),\tilde{\theta}_{t+1})\middle|\,\chi_t(\hat{\theta}^i_t,\theta^{-i}_t),\theta_t\right]\right|>\epsilon$ for all $\theta^i_t\neq\hat{\theta}^i_t$ and $\theta^{-i}_t\in\Theta^{-i}_t$.

Proof of Step 2 We prove it by mathematical induction. For $n=0$, we have already proved in Step 1. Assume that we have all the four conditions for $n-1$ $(n\geq 1)$. We will show that it is true for n.

Take any such t arbitrarily. As we have shown in the proof of Lemma 2, there exists $b^i_{t+1}:\Theta_t\to\Delta(\Theta_{t+1})$ s.t. $||b^i_{t+1}(\theta_t)||=1$ and defining

$$\phi^i_{t,t+1}(\theta_t,\theta_{t+1})=b^i_{t+1}(\theta_t)(\theta_{t+1})-\mathbb{E}\left[b^i_{t+1}(\theta_t)(\tilde{\theta}_{t+1})\middle|\,\chi_t(\theta_t),\theta_t\right], \tag{57}$$

and

$$\phi^i_{t,\tau}(\boldsymbol{\theta}^{\tau}_t)=\alpha^i_t(\theta_t,\theta^{-i}_{t+1})\phi^i_{t+1,\tau}(\boldsymbol{\theta}^{\tau}_t) \tag{58}$$

for all $\tau > t+2$, where

$$\begin{aligned}&\alpha_t^i(\theta_t, \theta_{t+1}^{-i})\\ &= \max_{\substack{\theta_{t+1}^i, \hat{\theta}_{t+1}^i \in \Theta_{t+1}^i \\ \theta_{t+1}^i \neq \hat{\theta}_{t+1}^i}} \frac{b_{t+1}^i(\theta_t)(\theta_{t+1}) - b_{t+1}^i(\theta_t)(\hat{\theta}_{t+1}^i, \theta_{t+1}^{-i})}{\delta \left| \mathbb{E}\left[\Phi_{t+1,t+2}^i((\hat{\theta}_{t+1}^i, \theta_{t+1}^{-i}), \tilde{\theta}_{t+2}) \middle| \chi_{t+1}(\hat{\theta}_{t+1}^i, \theta_{t+1}^{-i}), \theta_{t+1} \right] \right|},\end{aligned} \tag{59}$$

$(\phi_{t,\tau}^i)_{\tau=t+1}^\infty$ reveals the state in t costlessly.

Since $\phi_{t+1,\tau}^i(\boldsymbol{\theta}_t^\tau) = 0$ for all $\tau \geq t+n+1$, so $\phi_{t,\tau}^i$ is. Furthermore, $\phi_{t,t+1}^i(\theta_t, \theta_{t+1}) < 2$ for all $\theta_t \in \Theta_t$ and $\theta_{t+1} \in \Theta_{t+1}$, and $\phi_{t,\tau}^i(\boldsymbol{\theta}_t^\tau) < 2/\delta\epsilon$ holds for all $\boldsymbol{\theta}_t^\tau$. Thus, $(\phi_{t,\tau})_{\tau=t+1}^\infty$ is bounded by $\bar{\phi}_n = \max\{2, 2/\delta\epsilon\}$.

In addition, for all $t \in \mathbb{N}$, $x_t \in X_t$, $\theta_t \in \Theta_t$

$$\rho\left(\mu_{t+1}(x_t, \theta_t), \operatorname{conv}\{\mu_{t+1}(x_t, (\hat{\theta}_t^i, \theta_t^{-i}))\}_{\hat{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}\right) > k. \tag{60}$$

Therefore, in a similar way to the proof in Step 1, it is verified that

$$\left| \mathbb{E}\left[\Phi_{t,t+1}^i((\hat{\theta}_t^i, \theta_t^{-i}), \tilde{\theta}_{t+1}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t \right] \right| > \delta k = \epsilon. \ \blacksquare \tag{61}$$

Step 3 There exists a $((\phi_{s,\tau})_{\tau=s+1}^\infty)_{s=0}^\infty$ s.t.

(i) $(\phi_{s,\tau})_{\tau=s+1}^\infty$ is an appropriate costless revelation payment rule in s, i.e., it satisfies all the three conditions in Definition 4,

(ii) $((\phi_{s,\tau})_{\tau=s+1}^\infty)_{s=0}^\infty$ has a uniform upper bound $\bar{\phi} < \infty$,

(iii) $\phi_{s,\tau}^i(\boldsymbol{\theta}_s^\tau) = 0$ for all $\tau \geq s+T$,

(iv) there exists $\epsilon > 0$ s.t. $\left| \mathbb{E}\left[\Phi_{t,t+1}^i((\hat{\theta}_t^i, \theta_t^{-i}), \tilde{\theta}_{t+1}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t \right] \right| > \epsilon$ for all t, $\theta_t^i \neq \hat{\theta}_t^i$ and $\theta_t^{-i} \in \Theta_t^{-i}$.

Proof of Step 3 We will construct the entire $((\phi_{s,\tau})_{\tau=s+1}^\infty)_{s=0}^\infty$ in a following way. In period s,

(i) if for all $x_s \in X_s$ and $\theta_s \in \Theta_s$,

$$\rho\left(\mu_{s+1}^{-i}(x_s, \theta_s), \operatorname{conv}\{\mu_{s+1}^{-i}(x_s, (\hat{\theta}_s^i, \theta_s^{-i}))\}_{\hat{\theta}_s^i \in \Theta_s^i \setminus \{\theta_s^i\}}\right) > k, \tag{62}$$

we construct $(\phi_{s,\tau})_{\tau=s+1}^\infty$ in a way introduced in Step 1.

(ii) otherwise, we construct $(\phi_{s,\tau})_{\tau=s+1}^\infty$ in a way introduced in Step 2.

By assumption, there exists a period which satisfies the condition (i) for every T consecutive periods. Then, letting $\bar{\phi} = \max\{\bar{\phi}_0, ..., \bar{\phi}_T\}$, it is obvious that this $((\phi_{s,\tau})_{\tau=s+1}^\infty)_{s=0}^\infty$ satisfies all the four conditions above. $\blacksquare$

Step 4 Given this result, we will show that wp-EPIC payment rule ψ constructed in the proof of Theorem 4, has a uniform upper bound. Recall that the value of ψ_t^i is given by

$$\psi_t^i(\boldsymbol{\theta}^t) = -v_t^i(\chi_t(\theta_t), \theta_t) + \sum_{s=0}^{t-1} \alpha_s^i(\theta_s^{-i})\phi_{s,t}^i(\boldsymbol{\theta}_s^t). \tag{63}$$

where

$$\alpha_t^i(\theta_t^{-i}) = \max_{\substack{\theta_t^i, \hat{\theta}_t^i \in \Theta_t^i \\ \theta_t^i \neq \hat{\theta}_t^i}} \frac{-v_t^i(\chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t) + v_t^i(\chi_t(\hat{\theta}_t^i, \theta_t^{-i}), (\hat{\theta}_t^i, \theta_t^{-i}))}{\delta \left| \mathbb{E}\left[\Phi_{t,t+1}^i((\hat{\theta}_t^i, \theta_t^{-i}), \tilde{\theta}_{t+1}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t \right] \right|}. \tag{64}$$

Since v_t^i's are uniformly bounded, $|v_t^i(x_t, \theta_t)| < \bar{v}$ for all $i \in \mathcal{I}$, $t \in \mathbb{N}$, $x_t \in X_t$ and $\theta_t \in \Theta_t$. In addition, for all t, $\theta_t^i \neq \hat{\theta}_t^i$ and $\theta_t^{-i} \in \Theta_t^{-i}$, $\left| \mathbb{E}\left[\Phi_{t,t+1}^i((\hat{\theta}_t^i, \theta_t^{-i}), \tilde{\theta}_{t+1}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t \right] \right| > \epsilon$. Thus,

$$|\alpha_t^i| < \frac{2\bar{v}}{\delta\epsilon} \tag{65}$$

holds. Since $\phi_{s,t}^i(\boldsymbol{\theta}_s^t) = 0$ for all $t \geq s + T$, and there exists $\bar{\phi} \in \mathbb{R}$ s.t. $|\phi_{s,t}^i(\boldsymbol{\theta}_s^t)| < \bar{\phi}$ for all i, t, $\boldsymbol{\theta}_s^t$. Then,

$$|\psi_t^i| < \bar{v} + \frac{2T\bar{v}\bar{\phi}}{\delta\epsilon}. \tag{66}$$

Thus, this ψ is uniformly bounded. Applying Theorem 1, we obtain the desired result. ■