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On the uniqueness of Bertrand equilibrium

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Abstract

We introduce product differentiation in Dastidar's (1995) model of price competition with strictly convex costs. We find that even a small degree of product differentiation resolves the problem of the indeterminacy of equilibria. Although a continuum of equilibria exists in a homogeneous product market, the competitive price equilibrium is the only robust one. Specifically, as long as the equilibrium correspondence is nonempty, the equilibrium price converges to the competitive price when the degree of product differentiation shrinks to zero.

JEL classification numbers: C72, L13

Key words: price competition, strictly convex cost, uniqueness of equilibrium, increasing marginal costs, product differentiation

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1   Introduction

Bertrand competition is one of the most important models in the literature on oligopoly, and it has been intensively discussed in various contexts. In a homogeneous product market, when firms have identical constant marginal costs, Bertrand competition yields a unique equilibrium price that is equal to the perfect-competition price. However, this clear result does not hold in the case of strictly convex costs. Since Edgeworth (1897), many researchers have demonstrated that a pure strategy equilibrium does not exist under strictly convex costs. The non-existence of pure strategy equilibrium can be a serious obstacle in using Bertrand models. First, the property of mixed strategy equilibrium is generally complicated and it causes models that involve Bertrand competition to become intractable. Second, more importantly, it is doubtful whether decision makers in firms shoot dice for price selection.[1]

Dastidar (1995) introduces an assumption that a firm always supplies for all of the demands it faces, and shows the existence of pure strategy equilibrium in such a market.[2] However, his setting leads to another problem: the indeterminacy of equilibria. Specifically, a continuum of pure strategy equilibria exists in his model.

In this paper, we introduce product differentiation in Dastidar's (1995) model. We find that even a small degree of product differentiation resolves the problem of the non-uniqueness of equilibria in his model: the equilibrium price converges to the competitive price when the degree of product differentiation shrinks to zero. Our result highlights the utility of Dastidar's approach. Although a continuum of equilibria exists in his setting, we can focus on the Walrasian outcome because all of the other equilibria are unstable and vulnerable to product differentiation.

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[1]See Friedman (1988) and Martin (2001).

[2]Vives (1999, Chapter 5) explains the industries in which this assumption is plausible. Dastidar's (1995) model is extensively used in various contexts. See Abbink and Brandts (2008) for experimental approach, Ania (2008) for evolutionary game, Dastidar (1997, 2001) for competition policy implications, Hoernig (2002) for mixed strategy equilibria, and Ogawa and Kato (2006) for public policy and privatization.

2 The Model

Suppose that there are n symmetric firms in an industry. All firms have an identical cost function $C : \mathbb{R}_+ \to \mathbb{R}_+$, which is assumed to be increasing, continuously differentiable, and strictly convex. Without loss of generality, we assume $C(0) = C'(0) = 0$. In what follows, $C'(\cdot)$ and its inverse are denoted by $MC(\cdot)$ and $Y(\cdot)$, respectively.

We consider a sequence of markets in which the firms' products are differentiated, but the degree of production differentiation shrinks to zero. Then, we analyze the limit of equilibria in the differentiated markets. Let $d \in \mathbb{R}_+$ denote a degree of product differentiation: $d = 0$ means that the firms produce homogeneous commodities and $d > 0$ indicates that the products are differentiated. For each $d \in \mathbb{R}_+$, let $D_i^d : \mathbb{R}_+^n \to \mathbb{R}_+$ denote the contingent demand function for firm i under product differentiation d. We impose the following assumptions on D_i^d.

Assumption 1. For all $d > 0$, (a) D_i^d is continuous, and (b) D_i^d is decreasing in p_i and increasing in p_j $(j \neq i)$ at any point $(p_1, \ldots, p_n)$ such that $D_i^d(p_1, \ldots, p_n) > 0$.

Assumption 2. For all $d > 0$ and $(p_1, \ldots, p_n)$, if $p_i = p_j$, then $D_i^d(p_1, \ldots, p_n) = D_j^d(p_1, \ldots, p_n)$.

Assumption 3. For all $d > 0$, $D^d(p) := \sum_i D_i^d(p, \ldots, p)$ is decreasing in p.

Assumption 4. As $d \searrow 0$, D_i^d converges pointwise to D_i^0 defined by

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D_i^0(p_1, \ldots, p_n) = \begin{cases} D(p_i)/|\arg\min_j p_j| & \text{if } p_i = \min_j p_j, \\ 0 & \text{otherwise,} \end{cases}
$$

where $D : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and bounded function such that (a) $\exists P[p \geq P \Leftrightarrow D(p) = 0]$, and (b) D is decreasing on $[0, P]$.

Let us explain Assumptions 1–4. Assumption 1 represents our intended interpretation that $d > 0$ refers to differentiated products. Assumption 2 means that firms setting an identical price face an identical demand under any degree of production differentiation. Assumption 3 indicates, roughly speaking, that the aggregate demand is decreasing in price under any degree of production differentiation. Assumption 4 indicates that as $d \searrow 0$, the demand function converges to that of the

homogeneous product market analyzed by Dastidar (1995). In the homogeneous product market with D_i^0, the firm setting the lowest price obtain the entire demand. In case of a tie, the firms setting the lowest price obtain an equal share.[3]

Assumptions 1–4 imply the following two lemmata.

Lemma 1. *Suppose that $\{d_k\}_{k\in\mathrm{N}}$ is a sequence such that $d_k \to 0$. For any p and $\epsilon > 0$, there exist $\delta > 0$ and $K \in \mathbb{N}$ such that $|D_i^{d_k}(p,\ldots,p) - D_i^{d_k}(p',\ldots,p')| < \epsilon$ if $|p' - p| < \delta$ and $k \geq K$.*

Proof. Fix arbitrary p and ϵ, and take $a, b \in \mathbb{R}_+$ such that $p \in (a, b)$ and $D(a) - D(b) < n\epsilon$. By Assumptions 2 and 4, $D^{d_k}(a)$ and $D^{d_k}(b)$ converge to $D(a)$ and $D(b)$, respectively. Then, there exists $K \in \mathbb{N}$ such that $|D^{d_k}(a) - D^{d_k}(b)| < n\epsilon$ for all $k \geq K$. By Assumption 3, for all $p' \in (a, b)$ and $k \geq K$, $|D^{d_k}(p) - D^{d_k}(p')| < n\epsilon$, which is equivalent to $|D_i^{d_k}(p,\ldots,p) - D_i^{d_k}(p',\ldots,p')| < \epsilon$ by Assumption 2. ◻

Lemma 2. *Suppose that $\{(d_k, p_k)\}_{k\in\mathrm{N}}$ is a sequence such that $d_k \to 0$ and $p_k \to p$. If $D_i^0(p,\ldots,p)$ is strictly greater (resp. smaller) than $Y(p)$, then there exists $K \in \mathbb{N}$ such that $D_i^{d_k}(p_k,\ldots,p_k)$ is strictly greater (resp. smaller) than $Y(p_k)$ for all $k \geq K$.*

Proof. By Lemma 1 and the assumption that $p_k \to p$, it can be easily checked that as $k \to \infty$, $D_i^{d_k}(p_k,\ldots,p_k) \to D_i^0(p,\ldots,p)$. By the continuity of MC, it also follows that $Y(p_k) \to Y(p)$. These directly imply the statement. ◻

We formulate price competition games corresponding to each $d \in \mathbb{R}_+$ as follows. Following Dastidar (1995), we assume that a firm always supplies for all of the demands it faces even if that quantity is not optimal.[4] Specifically, given a degree of production differentiation d, i's payoff (or profit) function π_i^d is defined by $\pi_i^d = p_i D_i^d - C(D_i^d)$. Each firm i simultaneously and independently chooses $p_i \in \mathbb{R}_+$ so as to maximize π_i^d.

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[3]For a discussion on a more general sharing rule in homogeneous good markets, see Hoernig (2007).

[4]Note that when a firm has a strict convex cost, an increase in supply does not necessarily raise its profit.

3 Results

Let p^* be the competitive price in the limit homogeneous good market, i.e.,

$$Y(p^*) = D_i^0(p^*, \ldots, p^*). \tag{1}$$

Under the assumption that C is strictly convex, p^* uniquely exists.

We define *symmetric Nash equilibrium correspondence S* as follows: $p \in S(d)$ if and only if $(p, \ldots, p)$ is a Nash equilibrium in the price game under d. Dastidar (1995) shows that there exists a continuum of equilibrium when products are homogeneous, i.e., $S(0) = [p^m, p^M]$ for certain $p^m, p^M \in \mathbb{R}_{++}$. However, the following proposition states that a small degree of production differentiation excludes all equilibria, except for the competitive price one. In other words, S is not lower hemi-continuous at $d = 0$.

Proposition 1. *Under Assumptions 1–4, there is no sequence $\{(d_k, p_k)\}_{k \in \mathbb{N}}$ such that $d_k > 0$ and $p_k \in S(d_k)$ for all k, and $\lim_{k \to \infty} p_k = p \neq p^*$.*

Proof. By way of contradiction, assume that there exists such a sequence $\{(d_k, p_k)\}_{k \in \mathbb{N}}$.

First, suppose that $p \equiv \lim p_k < p^*$. Then, it follows from Lemma 2 that for sufficiently large k, $D_i^{d_k}(p_k, \ldots, p_k) > Y(p_k)$. Then, by Assumption 1, there exists $\hat{p} > p_k$ such that $D_i^{d_k}(\hat{p}; p_k, \ldots, p_k) = Y(\hat{p})$. That is, each firm can profitably deviate from $(p_k, \ldots, p_k)$ by charging $\hat{p}$, which is a contradiction to the assumption that $p_k \in S(d_k)$.

Second, suppose that $p \equiv \lim p_k > p^*$ and $D(p) < Y(p)$. Then, Lemma 1 implies that $\pi_i^{d_k}(p_k, \ldots, p_k) \to pD(p)/n - C(D(p)/n)$.[5] On the other hand, by Assumptions 1 and 4, for any $p' < p$, $D_i^{d_k}(p'; p_k, \ldots, p_k) \to D(p')$ and thus $\pi_i^{d_k}(p'; p_k, \ldots, p_k) \to p'D(p') - C(D(p'))$. Then, for p' that is sufficiently close to p, $\lim \pi_i^{d_k}(p_k, \ldots, p_k) < \lim \pi_i^{d_k}(p'; p_k, \ldots, p_k)$, which is a contradiction to the assumption that $p_k \in S(d_k)$ for all $k \in \mathbb{N}$.[6]

Finally, suppose that $p \equiv \lim p_k > p^*$ and $D(p) \geq Y(p)$. It follows that $D(p') > Y(p')$ for any $p' < p$. On one hand, since $D_i^{d_k}(p'; p_k, \ldots, p_k) \to D(p')$ as $k \to \infty$, $D_i^{d_k}(p'; p_k, \ldots, p_k) > Y(p')$ for

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[5]Note that by Assumption 2, $D_i^0(p, \ldots, p) = D(p)/n$.
[6]Note that $D(p') < Y(p')$ for p' that is sufficiently close to p.

sufficiently large k. On the other hand, by Lemma 2, $D_i^{d_k}(p_k, \ldots, p_k) < Y(p_k)$ for sufficiently large k. Thus, by Assumption 1, as long as k is sufficiently large, there exists $\hat{p}_k \in (p', p_k)$ such that $D_i^{d_k}(\hat{p}_k; p_k, \ldots, p_k) = Y(\hat{p}_k)$. Since p' is arbitrary, it can be easily checked that $\hat{p}_k \to p$. Therefore, $D_i^{d_k}(\hat{p}_k; p_k, \ldots, p_k) \to Y(p)$, and thus,

$$\lim \pi_i^{d_k}(\hat{p}_k; p_k, \ldots, p_k) = pY(p) - C(Y(p)) > \lim \pi_i^{d_k}(p_k, \ldots, p_k), \tag{2}$$

which is a contradiction to the assumption that $p_k \in S(d_k)$ for all $k \in \mathbb{N}$. $\square$

Proposition 1 states that the equilibrium price cannot converge to $p \neq p^*$ as $d \to 0$, whereas a continuum of equilibria exists when $d = 0$. Let us explain the intuition.

When $d = 0$, firm i's demand is discontinuous at a point $(p, \ldots, p)$, as shown in Figure 1. If firm i sets a slightly higher price than p, it does not obtain any demand. Thus, firm i has no incentive for upward deviation as long as its profit before deviation is non-negative. The condition of non-negative profit before deviation yields the lower bound p^m of equilibrium prices. If firm i sets a slightly lower price than p, it obtains all of the demands. However, since marginal cost is increasing, the deviation reduces profits, unless p is sufficiently high. This condition yields the upper bound p^M of equilibrium prices. Since it is obvious that $p^m < p^* < p^M$, a continuum of equilibria exists when $d = 0$.

When $d > 0$, firm i's demand is continuous. Even though the elasticity of demand is large at $(p, \ldots, p)$, it is finite, as seen in Figure 2. If $p < p^*$, each firm i supplies less than it wants to at $(p, \ldots, p)$. Thus, firm i can increase its profit by setting a slightly higher price than p by continuity, and a price below the Walrasian one cannot be an equilibrium under product differentiation. If $p > p^*$, i supplies more than it wants to at $(p, \ldots, p)$. Thus, by continuity, firm i can increase its profit by setting a slightly lower price than p when d is sufficiently close to zero but is strictly positive. Hence, the equilibrium price cannot converge to $p \neq p^*$ as $d \to 0$.

Furthermore, imposing another moderate assumption, we can show that the equilibrium price must converge to p^* as $d \searrow 0$ as long as we can choose a sequence of equilibrium prices.

Assumption 5. There exists P' such that $D^d(P') = 0$ for all $d > 0$.

Proposition 2. *Under Assumptions 1–5, if $\{(d_k, p_k)\}_{k \in \mathbb{N}}$ is a sequence such that $d_k > 0$ and $p_k \in S(d_k)$ for all k, and $\lim_{k \to \infty} d_k = 0$, then $p_k \to p^*$.*

Proof. First, suppose that $\liminf p_k < p^*$. Then, there exists a subsequence $\{(d_{k_i}, p_{k_i})\}_{i \in \mathbb{N}}$ such that $d_{k_i} \searrow 0$ and $p_{k_i} \to p < p^*$ as $i \to \infty$. However, this yields a contradiction as in the proof of Proposition 1.

Second, suppose that $\limsup p_k > p^*$. Note that Assumption 5 implies that p_k must be bounded and thus, $\limsup p_k < \infty$. Then, there exists a subsequence $\{(d_{k_i}, p_{k_i})\}_{i \in \mathbb{N}}$ such that $d_{k_i} \searrow 0$ and $p_{k_i} \to p > p^*$ as $i \to \infty$. This yields a contradiction as shown in the proof of Proposition 1.

We have shown that $\limsup p_k \leq p^* \leq \liminf p_k$. Since $\liminf p_k \leq \limsup p_k$ by definition, we can conclude that $\limsup p_k = \liminf p_k = p^*$. $\qquad\square$

Proposition 2 guarantees the robustness of the competitive equilibrium under $d = 0$, while Proposition 1 states that the other equilibria are fragile. These results highlight the significance of Dastidar's (1995) model. When we regard a homogeneous good market model as an approximation of a market with a small degree of product differentiation, the disadvantage of using Dastidar's model (i.e., equilibrium multiplicity) vanishes, and its advantage (i.e., existence of pure strategy equilibrium) is retained.

4   Concluding Remarks

In this paper, we introduce product differentiation in Dastidar's (1995) model of price competition with strictly convex costs. Dastidar (1995) discussed a Bertrand model where a firm always supplies for all of the demands it faces and showed that in a homogeneous good market, a continuum of equilibria exists. We find that even a small degree of product differentiation resolves the problem of the indeterminacy of equilibria. The equilibrium price converges to the competitive price equilibrium when the market becomes close to the homogeneous good market, and thus only one equilibrium is robust with respect to product differentiation.

Our result depends on the assumption that a firm always supplies for all of the demands it faces. If we discard this assumption, a pure strategy equilibrium does not exist in a homogeneous product

market, and a small degree of production differentiation does not yield a pure strategy equilibrium, although a large degree of differentiation usually yields a pure strategy equilibrium. This fact indicates that our result regarding the significance of a small degree of product differentiation is not obvious in a general context of price competition, although the intuition behind it is natural.

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