# Cost uncertainty and spatial agglomeration

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**Abstract**

We consider a linear city model à la Hotelling with price competition. We introduce cost uncertainty into the model. We consider a three-stage game: first, each firm chooses its location on the linear city; second, the costs of the firms are determined; and third, each firm sets its price. If the cost uncertainty is significant, each firm locates at the central point in an equilibrium outcome (minimum differentiation). If the cost uncertainty is insignificant, each firm locates at the edge of the linear city (maximum differentiation). For the middle range of cost uncertainty, both outcomes above could appear in equilibrium (multiple equilibria).

**JEL Classification Codes:** L13, D21, D43, R32

**Key Words:** uncertainty, agglomeration, linear city, *ex post* cutthroat competition.

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# 1 Introduction

Since the seminal work of Hotelling (1929), the model of spatial competition, which is one of the most important models of oligopoly, has been seen by many subsequent researchers as an attractive framework for analyzing product differentiation. The major advantage of this approach is that it allows an explicit analysis of product selection. Of particular interest is the equilibrium pattern of product locations and the degree of product differentiation. The original result of Hotelling (1929) is that firms produce similar products (minimum differentiation).

d'Aspremont *et al.* (1979), introduce price competition into the location model of Hotelling (1929) and consider two-stage location-price games. They show that there is no pure strategy equilibrium under original assumptions in Hotelling (1929) when transport costs are proportional to the distance between firm and consumer. They show that a pure strategy equilibrium exists when transport costs are quadratic. As opposed to the result in the original work of Hotelling (1929), in equilibrium, the products are maximally differentiated.

In this paper, we also consider a location model with price competition. We introduce a cost uncertainty into the duopoly model. The following three-stage game is discussed: first, each firm chooses its location on the linear city; second, the costs of the firms are determined; and third, each firm sets its price. We find that (i) spatial agglomeration (minimum differentiation) appears in equilibrium if the cost uncertainty is significant; (ii) maximum differentiation appears in equilibrium if the cost uncertainty is insignificant; and (iii) both maximum and minimum differentiations can be equilibrium outcomes if the cost uncertainty lies in the middle range.

In our model, firms dare to choose minimum differentiation if cost uncertainty is significant. Our model provides a good description of *ex post* aggressive competitions between firms facing uncertainty.

One possible source of *ex post* cost heterogeneity is a firm's nationality. For example, consider the competition between an EU firm and a US firm. Their cost differences depend on foreign exchange rates. If the dollar becomes strong relative to the euro, the EU firm has a cost advantage over the US firm. In this case, the EU firm can earn large profits by choosing the central position. On the other hand, if it chooses the central position, it cannot obtain positive profits when it has a cost disadvantage. In our model, each firm dares to take this risk when the possible *ex post* cost difference is significant. Due to the fluctuation of the foreign exchange rates, the possible *ex post* cost difference is significant between firms located in different countries, while it becomes relatively small between domestic firms. Our result indicates that, even under Bertrand competition, which yields severe competition in the cases of homogeneous product markets, it is possible that firms choose small product differentiation if firms are located in different countries. For example, in the semiconductor industry, Micron (a US company), Samsung (a Korean company), and Infineon (a German company) produce a similar product, DRAM (dynamic random access memory). Several Japanese firms used to produce DRAM; however, all but one firm exited the market and began to specialize in other differentiated products. Within the US, many other companies, such as Intel, left this market to produce differentiated products, and no major firm, except for Micron, produces DRAM. There is another DRAM producer, in Korea, Hynix, but it could not survive without government support. In the aviation industry, Airbus and Boeing compete and survive even though they produce similar products. On the other hand, Boeing and Lockheed (both US firms) produced similar products; however, Lockheed ultimately changed its strategy to specialize in military products. In the tire industry, Bridgestone (a Japanese firm), Michelin (a French firm), and Goodyear (a US firm) have similar survival strategies, while Firestone (a US firm) has merged with Bridgestone.

Another possible source of *ex post* cost heterogeneity is rapid technological changes. In the developing stages of new products, the firm with the lowest cost changes frequently. For example, in the semiconductor industry, several firms produce less-differentiated products, DRAM. In the earlier stages of this competition, the winners were the US firms (Intel, Motorola, and Texas Instruments), in the next two stages (64-kbit and 256-kbit stages) NEC was the winner, and Toshiba took over in the following stage (1-megabyte stage). None of these companies is currently in the first place. When products go through frequent generation changes, firms operate under uncertainty, without knowledge of which one will lead the industry with the lowest cost; our analysis predicts that the lesser degree of product differentiation appears under those situations and that aggressive competition yielding large deficits inevitably appears with a positive probability.

In our model, firms face identical cost structure with a positive probability. In this case, central agglomeration (minimum differentiation) yields the cutthroat competition, and no firm obtains positive profits with a positive probability. In real economies, this type of competition is frequently observed in the semiconductor industry. Under those circumstances, the firms could avoid such an excessive competition and could make profit if they would differentiate their products. Our results suggest that *ex post* unprofitable competition caused by the lesser differentiation is consistent with rational behavior. In our model, each firm dares to choose central location, because it might earn large profits if it takes the cost advantage (its cost is much lower than that of the rival).

This result is related to Cardon and Sasaki (1998). They investigate (non-spatial) patent competition and show that clustering in the technological choice can appear in equilibrium because clustering makes the profit of the winner become huge. However, our paper's driving force is quite different from that of them. In their model, they assume that only one firm obtains the patent for one technology, while in our model both firms can produce even when they choose to produces homogeneous products. In their model, clustering restricts competition, while in our model central agglomeration accelerates competition and central agglomeration reduces joint profits of firms.

Some studies have already shown that the minimum differentiation may appear in equilibrium. Price collusion after firms have made location choices is considered Friedman and Thisse (1993). Cooperation between firms is considered in the form of information exchange through communication Mai and Peng (1999). The mechanisms of inducing spatial agglomeration of these papers are completely different from those in our model. Note that these papers does not explain *ex post* cutthroat competition in homogeneous product markets.

This paper is closely related to De Palma *et al.* (1985). They show that sufficient heterogeneity between firms induces central agglomeration.[1] They introduce unobservable attributes in brand choice and consider situations in which central agglomeration does not induce the standard Bertrand competition with homogeneous products. In our paper, only the Hotelling type product differentiation is considered and other kinds of product differentiation are not considered. Thus, in contrast to their models, severe Bertrand competition with homogeneous goods appears when firms agglomerate. Nevertheless, in our model, firms dare to choose minimum differentiation. Furthermore, they do not obtain strategic complementarity of location choices (result (iii)), and they do not discuss the welfare implications of agglomeration. Established important contributions in this field are presented in Bester (1998). The concepts of vertical quality characteristics, asymmetric information of this quality between sellers and consumers, and a limited number of repeated purchases by consumers are introduced by Bester (1998).[2] He shows that, even though firms agglomerate, they can avoid cutthroat competition and they always obtain positive profits because of the signaling effect. He finds that, in equilibrium, central agglomeration appears if the number of repeated purchases is more than one and not too large, that maximum differentiation appears if the number of repeated purchases is more than one and not too small, and that both agglomeration and maximum differentiation can appears if the number of repeated purchases is middle. In his analysis, the key point is that unobservable quality reduces the incentives for differentiation, relaxing price competition. Our model formulation is quite different from his. First, there is no reliance on either asymmetric information or signaling effects. Imperfect information (or uncertainty) is introduced, but incompleteness of information (or asymmetric information) is not assumed. Second, in our model, central agglomeration is derived, even in a one-shot

[1] Heterogeneity is introduced into the Hotelling model Ziss (1993). *Ex ante* cost heterogeneity between incumbent and a new entrant is considered. The results of Ziss (1993) are quite different from ours. In Ziss (1993), central agglomeration never appears in equilibrium, and no pure strategy equilibrium exists if both firms choose location simultaneously. For a discussion of cost heterogeneity in a circular-city model, see also Schulz and Stahl (1985). For studies about introducing another dimension of product differentiation, see, e.g., Ma and Burgess (1993) and Ishibashi (2001).

[2] Information asymmetry is also considered in Boyer *et al.* (1994) and Boyer *et al.* (2003). In their model, the entrant firm does not have information about the incumbent firm.

game. Finally, the appearance in equilibrium of aggressive and unprofitable competition without product differentiation can be explained in our model, while, in their model, firms always make profits.

The paper is organized as follows. The model is formulated in Section 2, and the equilibrium outcome is investigated in Section 3. Welfare is discussed in Section 4. Section 5 is the conclusion. All proofs of the lemmas and propositions are presented in the Appendix.

# 2 The model

We formulate a duopoly model. We consider a model in which a linear city of length 1 lies in the abscissa of a line and consumers are uniformly distributed with density 1 along this interval. Suppose that firm $i$ ($i = 1, 2$) is located at point $l_i \in [0, 1]$. A consumer living at $y \in [0, 1]$ incurs a transport cost of $t(l_i - y)^2$ when she purchases the product from firm $i$. The consumers have unit demands, i.e., each consumes one or zero unit of the product. Each consumer derives a surplus from consumption (gross of price and transport costs) equal to $s$. We assume that $s$ is so large that every consumer consumes one unit of the product. Firms 1 and 2 produce the same physical product.

In the model, the unit cost of the product for each firm is $c_i$, which is determined randomly. After the firms locate at the line, Nature determines $c_1$ and $c_2$ independently and simultaneously. $c_i = 0$ with probability 1/2, and $c_i = c$ with probability 1/2 ($i \in \{1, 2\}$).

The game runs as follows. Each firm is risk-neutral and maximizes its own expected profits. In the first stage, each firm $i$ chooses its location $l_i \in [0, 1]$ simultaneously. In the second stage, each firm knows its own cost $c_i$ and its rival's cost $c_j$. In the third stage, each firm $i$ chooses its price $p_i \in [c_i, \infty)$ simultaneously.[3] For a consumer living at

$$x = \frac{l_1 + l_2}{2} + \frac{p_2 - p_1}{2t(l_2 - l_1)}, \tag{1}$$

[3] Choosing $p_i < c_i$ is weakly dominated by choosing $p_i = c_i$; so we assume that the lower bound of the price is its cost.

the total cost is the same at either of the two firms. Thus, the demand of firm 1, $D_1$, and that of firm 2, $D_2$, are given by

$$D_1(p_1, p_2, l_1, l_2) = \min\{\max(x, 0), 1\},\ D_2(p_1, p_2, l_1, l_2) = 1 - D_1(p_1, p_2, l_1, l_2). \tag{2}$$

The profit of firm $i$ is

$$\pi = (p_i - c_i)D_i. \tag{3}$$

First, we present a result about the third-stage subgames given $l_1$, $l_2$, $c_1$, and $c_2$. It is possible that firm 1 could be the monopolist (i.e., $D_1 = 1$) by limit pricing, if $c_1 = 0$ and $c_2 = c$ and the difference between the costs ware large. In this case, firm 2 would never become the monopolist regardless of $l_1$ and $l_2$. Needless to say, if the required limit price were too low, firm 1 would give up the position of monopolist. The following Lemmas 1 (i) and (ii) present the conditions under which firm 1 becomes the monopolist.

**Lemma 1** *(i) Suppose that $l_1 \leq l_2$ and $c_1 < c_2$. $D_1 = 1$ if and only if $c_2 - c_1 \geq t(l_2 - l_1)(4 - l_1 - l_2)$. In this case the profit of firm 1 is $c_2 - c_1 - t(l_2 - l_1)(2 - l_1 - l_2)$. (ii) Suppose that $l_1 > l_2$ and $c_1 < c_2$. $D_1 = 1$ if and only if $c_2 - c_1 \geq t(l_1 - l_2)(2 + l_1 + l_2)$. In this case, the profit of firm 1 is $c_2 - c_1 - t(l_1 - l_2)(l_1 + l_2)$.*

We now discuss the case in which $D_1 < 1$. Suppose that $l_1 \leq l_2$. From Lemma 1 (i) we consider the cases in which $c_2 - c_1 < t(l_2 - l_1)(4 - l_1 - l_2)$. The first-order conditions are

$$\frac{\partial \pi_1}{\partial p_1} = 0 \quad \Leftrightarrow \quad \frac{c_1 + p_2 - 2p_1 + t(l_2 - l_1)(l_2 + l_1)}{2t(l_2 - l_1)} = 0, \tag{4}$$

$$\frac{\partial \pi_2}{\partial p_2} = 0 \quad \Leftrightarrow \quad \frac{c_2 + p_1 - 2p_2 + t(l_2 - l_1)(2 - l_1 - l_2)}{2t(l_2 - l_1)} = 0. \tag{5}$$

The second order conditions are satisfied. These equations yield

$$p_1 = \frac{2c_1 + c_2 + t(l_2 - l_1)(2 + l_1 + l_2)}{3},\ p_2 = \frac{c_1 + 2c_2 + t(l_2 - l_1)(4 - l_1 - l_2)}{3}.$$

The quantity supplied by firm 1 is

$$D_1 \quad = \quad \frac{t(l_2 - l_1)(2 + l_1 + l_2) + c_2 - c_1}{6t(l_2 - l_1)}.$$

The profit functions are

$$\pi_1 = \frac{(c_2 - c_1 + t(l_2 - l_1)(2 + l_1 + l_2))^2}{18t(l_2 - l_1)}, \quad \pi_2 = \frac{(c_1 - c_2 + t(l_2 - l_1)(4 - l_1 - l_2))^2}{18t(l_2 - l_1)}.$$

If $c_1 = c_2 = 0$ or $c_1 = c_2 = c$, the profit functions are

$$\pi_1 = \frac{t(l_2 - l_1)(2 + l_1 + l_2)^2}{18}, \quad \pi_2 = \frac{t(l_2 - l_1)(4 - l_1 - l_2)^2}{18}. \tag{6}$$

Suppose that $l_2 < l_1$. From Lemma 1 (ii), we consider the cases where $c_2 - c_1 < t(l_1 - l_2)(2 + l_1 + l_2)$. From the symmetry of the linear city, we obtain

$$p_1 = \frac{2c_1 + c_2 + t(l_1 - l_2)(4 - l_1 - l_2)}{3}, \quad p_2 = \frac{c_1 + 2c_2 + t(l_1 - l_2)(2 + l_1 + l_2)}{3}.$$

The quantity supplied by firm 1 is

$$D_1 = \frac{t(l_1 - l_2)(4 - l_1 - l_2) + c_2 - c_1}{6t(l_1 - l_2)}.$$

The profit functions are

$$\pi_1 = \frac{(c_2 - c_1 + t(l_1 - l_2)(4 - l_1 - l_2))^2}{18t(l_1 - l_2)}, \quad \pi_2 = \frac{(c_1 - c_2 + t(l_1 - l_2)(2 + l_1 + l_2))^2}{18t(l_1 - l_2)}.$$

If $c_1 = c_2 = 0$ or $c_1 = c_2 = c$, the profit functions are

$$\pi_1 = \frac{t(l_1 - l_2)(4 - l_1 - l_2)^2}{18}, \quad \pi_2 = \frac{t(l_1 - l_2)(2 + l_1 + l_2)^2}{18}.$$

If the cost advantage of firm 1 is significant, it is optimal for firm 1 to set a price inducing monopoly by firm 1 in the third stage. If firm 1 chooses $l_1 = l_2$, firm 1 obtains the whole market by setting $p_1 = c_2$. If it chooses $l_1 \neq l_2$, firm 1 must charge a price that is strictly lower than $c_2$ so as to obtain the whole market. Thus, if firm 1 relocated, it would prefer minimum differentiation. On the other hand, if $l_1 = l_2$, firm 2's profit would drop to zero. Thus, if firm 2 relocated, it would have a strong incentive to avoid agglomeration, and it would want to be far away from firm 1 so as to avoid severe competition against the strong rival.

# 3 Equilibrium

We use subgame perfect Nash equilibrium as equilibrium concept. The game is solved by backward induction.

**Proposition 1** *The agglomeration of firms (minimum differentiation) appears in an equilibrium if and only if* $c \geq \underline{c} \equiv (9-\sqrt{31})t/4 \sim 0.858t$.

We now show the intuition behind the proposition. We explain the reason why firm 1 also prefers the central location given that $l_2 = 1/2$, when $c$ is sufficiently large. Without loss of generality, we assume $l_1 \in [0, 1/2]$.

Consider the subgame after the costs are realized. Suppose that firm 1 has the cost advantage. In this case, it is optimal for firm 1 to set a price inducing monopoly by firm 1. If firm 1 also chooses $l_1 = 1/2$, firm 1 obtains the whole market by setting $p_1 = c_2 (> c_1)$. If it chooses $l_1 < 1/2$, firm 1 must charge $p_1 = c_2 - \alpha$, where $\alpha$ is equal to the difference between the transport costs to firm 1 and to firm 2 for the consumer living at point 1 (the edge of the far side from firm 1). Thus, firm 1 prefers minimum differentiation. On the other hand, suppose that firm 1 does not have the cost advantage. If firm 1 chooses minimum differentiation, firm 1 earns zero profits. Thus, firm 1 has a strong incentive to avoid agglomeration and wants to be far away from firm 2 so as to avoid severe competition against the rival. The feature is similar to that in which both firms' costs are at the same level (discussed by d'Aspremont, Gabszewicz, and Thisse 1979).

There is a trade-off between choosing $l_1 = 0$ and choosing $l_1 = 1/2$. The latter is better if the realized position of firm 1 is good, while it is worse otherwise. First, consider the case in which firm 1 has the cost disadvantage. As $c$ increases, the profit of firm 1 becomes smaller, and, eventually, its profit becomes negligible even when it chooses $l_1 = 0$. Thus, the advantage of choosing $l_1 = 0$ becomes smaller. Next, consider the case in which both firms have identical costs. In this case, the profit of each firm does not depend on $c$. Finally, consider the case in which firm 1 has the cost advantage. As $c$ increases, the profit of firm 1 becomes larger. Under these conditions, an increase in $c$ makes the effect of the last case on the profit becomes more significant. Thus, firm 1 has a strong incentive to choose minimum differentiation when $c$ is sufficiently high.

From this discussion, we conjecture that the higher $c$ is, the more attractive the minimum differentiation becomes. The following proposition states that it is true.

**Proposition 2** *Maximum differentiation appears in an equilibrium if and only if* $c \leq \bar{c} \equiv t(81 - \sqrt{3(1928\sqrt{241} - 29269)})/18 \sim 2.025t$.

We now show that propositions 1 and 2 are not always mutually exclusive. For intermediate values of $c$, both maximum and minimum differentiation can appear in equilibrium.

**Proposition 3** $\bar{c} \geq \underline{c}$.

Propositions 1–3 imply that both maximum and minimum differentiation can be equilibrium outcomes if $c \in [\underline{c}, \bar{c}]$. The strategic complementarity of location choice yields this multiplicity of equilibria. The central location by firm 2 increases the incentive of central location by firm 1 and vice versa. We explain the intuition for this strategic complementarity.

As discussed above, when firm 2 locates at the center, it is beneficial for firm 1 to locate at the center when it has cost advantage. On the other hand, it is beneficial for it to locate at the edge when it has the the cost disadvantage. When $c$ is not too small, the profit of firm 1 locating at the edge becomes small, and the difference in its profit between locating at the center and at the edge becomes small. It reduces the benefit of firm 1 locating at the edge, and firm 1 also chooses the central location. Suppose that firm 2 moves from the central point to the other edge. The move increases the difference between firm 1's profit locating at the center and that locating at the edge; resulting in the increase in the benefit for locating at the edge. Therefore, firm 1 prefers the edge (center) if its rival also chooses the edge (center).

## 4 Welfare

In this section, we briefly make a welfare comparison between minimum and maximum product differentiation. Without loss of generality, we assume $l_1 \leq l_2$. Social surplus $W$ is given by

$$
\begin{aligned}
W &= s - c_1 D_1 - c_2(1 - D_1) - t\Big(\int_0^{D_1} (y - l_1)^2 dy + \int_{D_1}^1 (y - l_2)^2 dy\Big) \\
&= s - c_1 D_1 - c_2(1 - D_1) - t\Big((l_2 - l_1)D_1^2 + (l_1^2 - l_2^2)D_1 + \frac{1}{3} - l_2 + l_2^2\Big). \qquad (7)
\end{aligned}
$$

Following two propositions state that each firm's incentive for central agglomeration is insufficient from the viewpoint of social welfare, while it is excessive from the viewpoint of total profits of firms.

**Proposition 4** *The expected total social surplus is larger in minimum differentiation than in maximum differentiation.*

**Proposition 5** *(i) The expected total profit at maximum differentiation is larger than or equal to that at minimum differentiation and (ii) it is strictly larger if and only if $c < 3t$.*

If $c > \bar{c}$, the equilibrium outcome is the minimum differentiation while the joint profits are larger at the maximum differentiation. This indicates that firms are faced with a "prisoners' dilemma". Both firms prefer maximum differentiation to minimum differentiation, but each firm has an incentive for reducing product differentiation by choosing the central location, which results in the reduction of total profits. Furthermore, Proposition 4 indicates that such a competition is beneficial from the viewpoint of social surplus. Central agglomeration increases the supply of the more efficient firm and reduces the total production costs.[4]

[4] Although welfare-improving production substitution is rarely discussed in the context of a location-price model, some works discuss it in completely different contexts. See Lahiri and Ono (1988) in the context of the Cournot competition, Riordan (1998) in the context of vertical foreclosure, Matsumura (1998) in the context of mixed markets, and Ono (1990) Lahiri and Ono (1998) in the context of international trade and direct investment.

## 5 Concluding remarks

In the paper, we introduce a cost uncertainty into the Hotelling model. We assume that each firm chooses its location before observing its own and the rival's costs and chooses its price after observing them. We think that this situation is realistic. For example, consider the fluctuation of foreign exchange rates. It is natural to assume that it is difficult for a firm to change its product responding to changes in the exchange rate, while it is relatively easy for a firm to change its price responding to the exchange rate.

In the model, we find that the significant cost heterogeneity yields minimum product differentiation. Firms are faced with a severe Bertrand competition, and, with a positive probability, neither firm will make a profit. If they produce differentiated goods products, they will always make a profit. Nevertheless, they dare to produce homogeneous products because the winner obtains large profits with a positive probability.

Next, we find that if the cost uncertainty is insignificant, the equilibrium outcome is the maximum differentiation. If the cost uncertainty lies in the middle range, both minimum and maximum differentiation become equilibrium location patterns (multiple equilibria). We also find that each firm's incentive for central agglomeration is insufficient from the viewpoint of social welfare, while it is excessive from the viewpoint of joint-profit maximization.

Finally, we make a remark on the robustness of our results. In this paper we assume that the correlation between two firms is zero. If it is negative (positive), $\underline{c}$ decreases (increases). In other words, the minimum (maximum) differentiation appears in equilibrium more easily under negative (positive) correlation. If the costs of the firms are perfectly correlated, no *ex post* heterogeneity between firms exists, and minimum differentiation never appears in equilibrium.

Using the model, we might be able to investigate the relation between strategies of product differentiation and cost-reducing R&D investments. We think that the result of the paper presents a hint to consider the relation which is a considerable future research.

## Appendix

**Proof of Lemma 1 (i):** We suppose that $l_1 \leq l_2$. The quantity supplied by firm 1, $D_1$, is

$$D_1(p_1, p_2, l_1, l_2) = \min\{\max(x, 0), 1\}, \text{ where } x = \frac{l_1 + l_2}{2} + \frac{p_2 - p_1}{2t(l_2 - l_1)}. \tag{8}$$

Given the locations of the firms, the first order derivatives are:

$$\begin{aligned} \frac{\partial \pi_1}{\partial p_1} &= \frac{c_1 + p_2 - 2p_1 + t(l_2 - l_1)(l_2 + l_1)}{2t(l_2 - l_1)}, \\ \frac{\partial \pi_2}{\partial p_2} &= \frac{c_2 + p_1 - 2p_2 + t(l_2 - l_1)(2 - l_1 - l_2)}{2t(l_2 - l_1)}. \end{aligned}$$

$D_1 = 1$ if and only if the following conditions are satisfied:

$$p_1 = c_2 - t(l_2 - l_1)(2 - l_1 - l_2), \quad p_2 = c_2, \tag{9}$$

$$\left.\frac{\partial \pi_1}{\partial p_1}\right|_{p_1 = c_2 - t(l_2 - l_1)(2 - l_1 - l_2),\ p_2 = c_2} \leq 0, \quad \left.\frac{\partial \pi_2}{\partial p_2}\right|_{p_1 = c_2 - t(l_2 - l_1)(2 - l_1 - l_2),\ p_2 = c_2} \leq 0,$$

$$\Leftrightarrow \quad \frac{c_1 + c_2 - 2(c_2 - t(l_2 - l_1)(2 - l_1 - l_2)) + t(l_2 - l_1)(l_2 + l_1)}{2t(l_2 - l_1)} \leq 0, \tag{10}$$

$$\frac{c_2 + c_2 - t(l_2 - l_1)(2 - l_1 - l_2) - 2c_2 + t(l_2 - l_1)(2 - l_1 - l_2)}{2t(l_2 - l_1)} \leq 0. \tag{11}$$

$p_1$ in (9) is the highest value for sustaining $D_1 = 1$, given that $p_2 = c_2$. If the inequalities in (10) and (11) are satisfied, none of the firms has an incentive to raise his price, that is, the prices in (9) are an equilibrium outcome. The left-hand side in (11) is zero, so (11) is satisfied. From (10), we have that $D_1 = 1$ if and only if $c_2 - c_1 \geq t(l_2 - l_1)(4 - l_1 - l_2)$. Q.E.D.

**Proof of Lemma 1 (ii):** We suppose that $l_1 > l_2$. The quantity supplied by firm 1, $D_1$, is

$$D_1(p_1, p_2, l_1, l_2) = \min\{\max(x, 0), 1\}, \text{ where } x = \frac{2 - l_1 - l_2}{2} + \frac{p_1 - p_2}{2t(l_1 - l_2)}. \tag{12}$$

Given the locations of the firms, the first derivatives are:

$$\begin{aligned} \frac{\partial \pi_1}{\partial p_1} &= \frac{c_1 + p_2 - 2p_1 + t(l_1 - l_2)(2 - l_2 - l_1)}{2t(l_1 - l_2)}, \\ \frac{\partial \pi_2}{\partial p_2} &= \frac{c_2 + p_1 - 2p_2 + t(l_1 - l_2)(l_1 + l_2)}{2t(l_1 - l_2)}. \end{aligned}$$

$D_1 = 1$ if and only if the conditions are satisfied:

$$p_1 = c_2 - t(l_1 - l_2)(l_1 + l_2), \quad p_2 = c_2, \tag{13}$$

$$\left.\frac{\partial \pi_1}{\partial p_1}\right|_{p_1=c_2-t(l_1-l_2)(l_1+l_2),\ p_2=c_2} \leq 0, \quad \left.\frac{\partial \pi_2}{\partial p_2}\right|_{p_1=c_2-t(l_1-l_2)(l_1+l_2),\ p_2=c_2} \leq 0,$$

$$\Leftrightarrow \quad \frac{c_1 + c_2 - 2(c_2 - t(l_1 - l_2)(l_1 + l_2)) + t(l_1 - l_2)(2 - l_2 - l_1)}{2t(l_1 - l_2)} \leq 0, \tag{14}$$

$$\frac{c_2 + c_2 - t(l_1 - l_2)(l_1 + l_2) - 2c_2 + t(l_1 - l_2)(l_1 + l_2)}{2t(l_1 - l_2)} \leq 0. \tag{15}$$

$p_1$ in (13) is the highest value for sustaining $D_1 = 1$, given that $p_2 = c_2$. If the inequalities in (14) and (15) are satisfied, none of the firms has an incentive to raise its price, that is, the prices in (13) are an equilibrium outcome. The left-hand side in (15) is zero, so it is satisfied. From (14) we have that $D_1 = 1$ if and only if $c_2 - c_1 \geq t(l_1 - l_2)(2 + l_1 + l_2)$. **Q.E.D.**

**Proof of Proposition 1:** Suppose that the strategy of firm 2 is $l_2 = 1/2$ in the first stage. We show that the best response of firm 1 is choosing $l_1 = 1/2$ if and only if $c \geq (9 - \sqrt{31})t/4$. Because of the symmetry of the linear city, we assume $l_1 \in [0, 1/2]$ without loss of generality.

In the first stage, each firm takes into account the following four possibilities; $c_1 = c_2 = 0$, $c_1 = c_2 = c$, $c_1 = 0$ and $c_2 = c$, and $c_1 = c$ and $c_2 = 0$. When $c_1 = c_2$, the profit of firm 1 is given by (6). To obtain the expected profit of firm 1, we must consider what happens when $c_1 \neq c_2$. As we can see from Lemma 1, we must consider whether or not the following two inequalities are satisfied (note that we now consider the case in which the strategy of firm 2 is $l_2 = 1/2$):

$$t(1/2 - l_1)(7/2 - l_1) \leq c, \quad \text{(Lemma 1 (i))}, \tag{16}$$

$$t(1/2 - l_1)(5/2 + l_1) \leq c, \quad \text{(Lemma 1 (ii))}. \tag{17}$$

The inequality in (16) is the condition that the quantity supplied by firm 1 is 1, if $c_1 < c_2$. The inequality in (17) is the condition that the quantity supplied by firm 1 is 0, if $c_2 < c_1$.

The inequality in (16) is satisfied if and only if

$$2 - \sqrt{\frac{9}{4} + \frac{c}{t}} \leq l_1 \leq 2 + \sqrt{\frac{9}{4} + \frac{c}{t}}. \tag{18}$$

The inequality in (17) is always satisfied regardless of $l_1$ if $c > 9t/4$. Otherwise, it is satisfied if and only if

$$l_1 \leq -1 - \sqrt{\frac{9}{4} - \frac{c}{t}} (< 0), \text{ or } -1 + \sqrt{\frac{9}{4} - \frac{c}{t}} \leq l_1. \tag{19}$$

From inequalities in (18) and (19), we have that both inequalities in (16) and (17) are satisfied for all $l_1 \in [0, 1/2]$ if $c \geq 7t/4$.

First, we assume that $c < 7t/4$. Since $0 \leq l_1 \leq 1/2$, $t(1/2-l_1)(5/2+l_1) < t(1/2-l_1)(7/2-l_1)$. Thus, if the inequality in (16) holds, the inequality in (17) is also satisfied. From (18) and (19), we divide the range of $l_1 \in [0, 1/2]$ into the following three segments:[5] (i) $l_1 \in [2 - \sqrt{\frac{9}{4} + \frac{c}{t}}, \frac{1}{2}]$ (both (16) and (17) are satisfied), (ii) $l_1 \in (\max\{0, -1 + \sqrt{\frac{9}{4} - \frac{c}{t}}\}, 2 - \sqrt{\frac{9}{4} + \frac{c}{t}})$ (only (17) is satisfied), and (iii) $l_1 \in [0, \max\{0, -1 + \sqrt{\frac{9}{4} - \frac{c}{t}}\}]$. (neither (16) nor (17) is satisfied).

(i) $l_1 \in \left[2 - \sqrt{\frac{9}{4} + \frac{c}{t}}, \frac{1}{2}\right]$: the inequalities in (16) and (17) are satisfied.

Since $2 - \sqrt{\frac{9}{4} + \frac{c}{t}} < 1/2$, range (i) is never empty. We show that for this range optimal location of firm 1 is 1/2. Under these conditions, from (16), we have that $D_1 = 1$ when $c_1 < c_2$. From (17), we have that $D_1 = 0$ when $c_1 > c_2$. Thus, the expected profit of firm 1 is given by

$$\begin{aligned} E[\Pi_1(l_1, l_2)] &= \frac{1}{4} \times \frac{t(l_2 - l_1)(2 + l_1 + l_2)^2}{18} + \frac{1}{4} \times \frac{t(l_2 - l_1)(2 + l_1 + l_2)^2}{18} \\ &\quad + \frac{1}{4} \times (c - t(l_2 - l_1)(2 - l_1 - l_2)). \end{aligned} \tag{20}$$

The first and the second terms are the expected profits in which both firms' costs are the same. The third term is the expected profit in which $c_1 = 0$ and $c_2 = c$. Substituting $l_2 = 1/2$ into $E[\Pi_1(l_1, l_2)]$ in (20) and differentiating it with respect to $l_1$, we have

$$\frac{\partial E[\Pi_1(l_1, 1/2)]}{\partial l_1} = \frac{t(1 - 2l_1)(19 + 2l_1)}{48} \geq 0, \tag{21}$$

for any $l_1$ belonging to this range and the equality holds only when $l_1 = 1/2$. Thus $l_1 = 1/2$ is the best for this range.

[5] We need not care about where each boundary of these three ranges belongs to, because the profit function is continiuous.

(ii) $l_1 \in \left(\max\{0, -1+\sqrt{\frac{9}{4}-\frac{c}{t}}\}, 2-\sqrt{\frac{9}{4}+\frac{c}{t}}\right)$: the inequality in (17) holds but the inequality in (16) is not satisfied.

Since $c \leq 7t/4$, $2-\sqrt{\frac{9}{4}+\frac{c}{t}} > 0$. We show that the optimal $l_1$ never lies in this range. Under these conditions, from (16), we have $D_1 < 1$ even if $c_1 < c_2 = c$. From (17), $D_1 = 0$ if $c_1 > c_2$. Then the expected profit of firm 1 is given by

$$\begin{aligned} E[\Pi_1(l_1, l_2)] &= \frac{1}{4} \times \frac{t(l_2-l_1)(2+l_1+l_2)^2}{18} + \frac{1}{4} \times \frac{t(l_2-l_1)(2+l_1+l_2)^2}{18} \\ &\quad + \frac{1}{4} \times \frac{(c+t(l_2-l_1)(2+l_1+l_2))^2}{18t(l_2-l_1)}. \end{aligned} \tag{22}$$

The first and the second terms are the expected profits in which both firms' costs are the same. The third term is the expected profit in which $c_1 = 0$ and $c_2 = c$. Substituting $l_2 = 1/2$ into $E[\Pi_1(l_1, l_2)]$ in (22) and differentiating it with respect to $l_1$, we have

$$\frac{\partial E[\Pi_1(l_1, 1/2)]}{\partial l_1} = \frac{\phi(l_1)}{288t(1-2l_1)^2}, \tag{23}$$

where $\phi(l_1) \equiv 16c^2 + 8ct(1-2l_1)^2 - 9t^2(1-2l_1)^2(1+2l_1)(5+2l_1)$. Differentiating $\phi(l_1)$, we have

$$\phi'(l_1) = 8t(1-2l_1)(9t - 4c + 72tl_1 + 36tl_1^2) > 0, \tag{24}$$

since we assume that $c < 7t/4$ and that $0 < l_1 < 1/2$ for range (ii). Therefore, one of the following three conditions must be satisfied. (a) $\phi \leq 0$ for all $l_1$ in range (ii), (b) $\phi < 0$ if and only if $l_1 < \tilde{l}_1$, where $l_1 < \tilde{l}_1$ is derived from $\phi(\tilde{l}_1) = 0$, or (c) $\phi \geq 0$ for all $l_1$ in range (ii). In case (a), the expected profit of firm 1 is non-increasing in $l_1$, in case (b), it is U-shaped, and in case (c), it is non-decreasing in $l_1$. In all cases, $l_1$ lying on range (ii) is dominated by either $l_1 = \max\{0, -1+\sqrt{\frac{9}{4}-\frac{c}{t}}\}$ or $l_1 = 2-\sqrt{\frac{9}{4}+\frac{c}{t}}$, since $E[\Pi_1(l_1, 1/2)]$ is continuous and maximized at one of the corners.

(iii) $l_1 \in \left[0, \max\{0, -1+\sqrt{\frac{9}{4}-\frac{c}{t}}\}\right]$: neither of the inequalities in (16) and (17) holds.

We show that $l_1 = 0$ is best for firm 1 in range (iii). If $-1+\sqrt{\frac{9}{4}-\frac{c}{t}} \leq 0$, it is obvious since no other alternative exists for this range. We assume that $-1+\sqrt{\frac{9}{4}-\frac{c}{t}} > 0$ ($c < 5t/4$) and show that the optimal $l_1$ is 0. From (16), we have $D_1 < 1$ when $c_1 < c_2$. From (17), $D_1 > 0$ when $c_1 > c_2$. The expected profit of firm $i$ is given by

$$\begin{aligned} E[\Pi_1(l_1, l_2)] \; = \; & \frac{1}{4} \times \frac{t(l_2 - l_1)(2 + l_1 + l_2)^2}{18} + \frac{1}{4} \times \frac{t(l_2 - l_1)(2 + l_1 + l_2)^2}{18} \\ + \; & \frac{1}{4} \times \frac{(c + t(l_2 - l_1)(2 + l_1 + l_2))^2}{18t(l_2 - l_1)} + \frac{1}{4} \times \frac{(t(l_2 - l_1)(2 + l_1 + l_2) - c)^2}{18t(l_2 - l_1)}. \end{aligned} \quad (25)$$

The first and the second terms are the expected profits in which both firms' costs are the same. The third term is the expected profit in which $c_1 = 0$ and $c_2 = c$. The fourth term is the expected profit in which $c_1 = c$ and $c_2 = 0$. Substituting $l_2 = 1/2$ into $E[\Pi_1(l_1, l_2)]$ in (25) and differentiating it with respect to $l_1$, we have

$$\frac{\partial E[\Pi_1(l_1, 1/2)]}{\partial l_1} = \frac{\psi(l_1)}{72t(1 - 2l_1)^2}, \quad (26)$$

where $\psi(l_1) \equiv 8c^2 - 3t^2(1 - 2l_1)^2(1 + 2l_1)(5 + 2l_1)$. Differentiating $\psi(l_1)$, we have

$$\psi'(l_1) = 24t^2(1 - 2l_1)(1 + 8l_1 + 4l_1^2) \geq 0. \quad (27)$$

Substituting $l_1 = -1 + \sqrt{9/4 - c/t}$ (the right corner of range (iii)) into $\psi(l_1)$, we have (note that we have assumed $-1 + \sqrt{9/4 - c/t} > 0$ ($c < 5t/4$)):

$$\psi\left(-1 + \sqrt{9/4 - c/t}\right) = 4(2c(18t - 5c) - 12c\sqrt{t(9t - 4c)}) < 0. \quad (28)$$

Therefore, $\psi(l_1) < 0$ for all $l_1$ lying on range (iii). This implies that expected profit of firm 1 is decreasing in $l_1$.

From the above discussions for the three ranges, we have that either $l_1 = 1/2$ or $l_1 = 0$ is best reply of firm 1. The best response of firm 1 is $l_1 = 1/2$ if and only if $E[\Pi_1(1/2, 1/2)] \geq E[\Pi_1(0, 1/2)]$. The condition is

$$\frac{c}{4} - \frac{8c^2 + 25t^2}{144t} \geq 0 \quad \Leftrightarrow \quad c \geq \frac{(9 - \sqrt{31})t}{4} \sim 0.858t.$$

Next, we assume that $7t/4 \leq c$. In this case, as mentioned above, both inequalities in (16) and (17) are satisfied for all $l_1 \in [0, 1/2]$. And from exactly the same discussion in range (i) above, we have that the firm 1's payoff is increasing in $l_1$ for $l_1 \in [0, 1/2)$. Thus firm 1's best reply is $l_1 = 1/2$.

**Q.E.D.**

**Proof of Proposition 2:** The structure of the proof of Proposition 2 is almost the same as Proposition 1. However, the proof is a little bit complicated because in the proof of Proposition 1 we can restrict $l_1 \in [0, 1/2]$ by symmetry, while we must consider $l_1 \in [0, 1]$ in this proof. Since the proof is complicated, we use six supplementary lemmas in the proof.

Suppose that the strategy of firm 2 is $l_2 = 1$ in the first stage. When $c_1 = c_2$, the profit of firm 1 is given by (6). To obtain the expected profit of firm 1, we must consider what happens when $c_1 \neq c_2$. As we can see from Lemma 1, we must consider whether or not the following two inequalities are satisfied (note that we now consider the case in which the strategy of firm 2 is $l_2 = 1$):

$$t(1 - l_1)(3 - l_1) \leq c, \quad \text{(Lemma 1 (i))}, \tag{29}$$

$$t(1 - l_1)(3 + l_1) \leq c, \quad \text{(Lemma 1 (ii))}. \tag{30}$$

The inequality in (29) is the condition under which the quantity supplied by firm 1 is 1 when $c_1 < c_2$. The inequality in (30) is the condition that the quantity supplied by firm 1 is 0 when $c_2 < c_1$.

Since $0 \leq l_1 \leq 1$, $t(1 - l_1)(3 - l_1) < t(1 - l_1)(3 + l_1)$. If the inequality in (30) holds, the inequality in (29) also holds. The inequality in (29) holds if and only if

$$2 - \sqrt{1 + \frac{c}{t}} \leq l_1 \leq 2 + \sqrt{1 + \frac{c}{t}}. \tag{31}$$

The inequality in (30) is always satisfied regardless of $l_1$ if $c > 4t$. If $c \leq 4t$, the inequality in (30) holds if and only if

$$l_1 \leq -1 - \sqrt{4 - \frac{c}{t}}, \text{ or } -1 + \sqrt{4 - \frac{c}{t}} \leq l_1. \tag{32}$$

The first inequality in (32) is never satisfied since $l_1 \geq 0$. Thus we can ignore it. From the inequalities in (32) and (31), we have that both inequalities in (29) and (30) are satisfied for all $l_1 \in [0, 1]$ if $c \geq 3t$.

We now show that the best response of firm 1 is $l_1 = 0$ if $c \leq t(81 - \sqrt{3(1928\sqrt{241} - 29269)})/18 \sim 2.025t$.

First, we consider the cases in which $c < 3t$. Since $c < 3t$, $2 - \sqrt{1 + \frac{c}{t}} < -1 + \sqrt{4 - \frac{c}{t}}$. From (32) and (31), we divide the range of $l_1 \in [0, 1]$ into the following three segments: (i-2) $l_1 \in \left(-1 + \sqrt{4 - \frac{c}{t}}, 1\right]$ (both (29) and (30) are satisfied); (ii-2) $l_1 \in \left(2 - \sqrt{1 + \frac{c}{t}}, -1 + \sqrt{4 - \frac{c}{t}}\right)$ (only (30) is satisfied); and (iii-2) $l_1 \in \left[0, 2 - \sqrt{1 + \frac{c}{t}}\right]$ (neither (29) nor (30) is satisfied).

We now present three Lemmas showing the optimal location of firm 1 for each of three range above.

**Lemma A1:** *Suppose that $l_2 = 1$. Suppose that $c < 3t$. (i) Among $l_1$ lying on the range (i-2), $l_1 = \frac{\sqrt{241}-14}{3}$ maximizes the expected profit of firm 1, if $c \geq 2(11\sqrt{241} - 163)t/9 \sim 1.726t$; and (ii) the expected profit of firm 1 is decreasing in $l_1$ for all $l_1$ belonging to the range (i-2) if $c < 2(11\sqrt{241} - 163)t/9 \sim 1.726t$.*

**Proof:** Since $-1 + \sqrt{4 - \frac{c}{t}} < 1$, the range (i-2) is never empty. From (29), we have that $D_1 = 1$ when $c_1 < c_2$. From (30), we have that $D_1 = 0$ when $c_2 < c_1$. Thus, the expected profit of firm 1 is given by

$$\begin{aligned} E[\Pi_1(l_1, l_2)] &= \frac{1}{4} \times \frac{t(l_2 - l_1)(2 + l_1 + l_2)^2}{18} + \frac{1}{4} \times \frac{t(l_2 - l_1)(2 + l_1 + l_2)^2}{18} \\ &\quad + \frac{1}{4} \times (c - t(l_2 - l_1)(2 - l_1 - l_2)). \end{aligned} \tag{33}$$

The first and the second terms are the expected profits in which both firms' costs are the same. The third term is the expected profit in which $c_1 = 0$ and $c_2 = c$. Substituting $l_2 = 1$ into $E[\Pi_1(l_1, l_2)]$ in (33) and differentiating it with respect to $l_1$, we have

$$\frac{\partial E[\Pi_1(l_1, 1)]}{\partial l_1} = \frac{t(15 - 28l_1 - 3l_1^2)}{36}. \tag{34}$$

We find that

$$\frac{\partial E[\Pi_1(l_1, 1)]}{\partial l_1} = (>, <)0 \quad \Leftrightarrow \quad l_1 = (<, >)\frac{\sqrt{241} - 14}{3} \sim 0.5081. \tag{35}$$

If $c \geq 2(11\sqrt{241} - 163)t/9 \sim 1.726t$, then $\frac{\sqrt{241}-14}{3} \geq -1 + \sqrt{4 - c/t}$ (the left corner of the range (i-2)). Therefore, if $c \geq 2(11\sqrt{241} - 163)t/9$, $l_1 = \frac{\sqrt{241}-14}{3}$ is the best response of firm 1. Otherwise, the expected profit is decreasing in $l_1$ for this range (so $l_1 = -1 + \sqrt{4 - c/t}$ is firm

1's best response). **Q.E.D.**

**Lemma A2:** *Suppose that $l_2 = 1$. Suppose that $c < 3t$. (i) the expected profit of firm 1 is non-increasing in $l_1$ for all $l_1$ belonging to the range (ii-2) if $c \leq \frac{2(491-26\sqrt{331})t}{25} \sim 1.438t$; (ii) among $l_1$ lying on the range (ii-2), neither of the two corners of this range maximizes the expected profit of firm 1, if $\frac{2(491-26\sqrt{331})t}{25} < c < \frac{2(11\sqrt{241}-163)t}{9}$; and (iii) The expected profit of firm 1 is non-decreasing in $l_1$ for all $l_1$ belonging to the range (ii-2) if $c \geq \frac{2(11\sqrt{241}-163)t}{9} \sim 1.726t$.*

**Proof:** From (29), we have that $D_1 = 1$ when $c_1 < c_2$. From (30), we have that $D_1 > 0$ when $c_2 < c_1$. Thus, the expected profit of firm 1 is given by

$$\begin{aligned} E[\Pi_1(l_1, l_2)] \; = \; & \frac{1}{4} \times \frac{t(l_2 - l_1)(2 + l_1 + l_2)^2}{18} + \frac{1}{4} \times \frac{t(l_2 - l_1)(2 + l_1 + l_2)^2}{18} \\ & + \frac{1}{4} \times (c - t(l_2 - l_1)(2 - l_1 - l_2)) + \frac{1}{4} \times \frac{(t(l_2 - l_1)(2 + l_1 + l_2) - c)^2}{18t(l_2 - l_1)}. \end{aligned} \quad (36)$$

The first and the second terms are the expected profits in which both firms' costs are the same. The third term is the expected profit in which $c_1 = 0$ and $c_2 = c$. The fourth term is the expected profit in which $c_1 = c$ and $c_2 = 0$. Substituting $l_2 = 1$ into $E[\Pi_1(l_1, l_2)]$ in (36) and differentiating it with respect to $l_1$, we have

$$\frac{\partial E[\Pi_1(l_1, 1)]}{\partial l_1} = \frac{c^2 - 2ct(1 - l_1)^2 + 3t^2(1 - l_1)^2(9 - 22l_1 - 3l_1^2)}{72t(1 - l_1)^2}. \quad (37)$$

We denote the numerator of the fraction as $\phi_2(l_1)$:

$$\phi_2(l_1) \equiv c^2 - 2ct(1 - l_1)^2 + 3t^2(1 - l_1)^2(9 - 22l_1 - 3l_1^2). \quad (38)$$

Note that the expected profit is decreasing (increasing) in $l_1$ if $\phi_2$ is negative (positive).

Differentiating $\phi_2(l_1)$, we have

$$\phi_2'(l_1) = 4t(1 - l_1)(c - 30t + 45tl_1 + 9tl_1^2). \quad (39)$$

Differentiating $\phi_2'(l_1)$, we have

$$\phi_2''(l_1) = 4t(75t - c - 72tl_1 - 27tl_1^2). \quad (40)$$

$\phi_2''(l_1)$ is decreasing in $l_1 \in [0,1]$. Since $\phi_2'$ is concave for this range, either of the two corners of this range minimizes $\phi_2'$. Substituting $l_1 = 2 - \sqrt{1+c/t}$ and $l_1 = -1 + \sqrt{4-c/t}$ (the corners of the range) into $\phi_2'(l_1)$, we have:

$$\begin{aligned} \phi_2'\left(2 - \sqrt{1+\frac{c}{t}}\right) &= 4(\sqrt{t(t+c)} - t)(10c + 105t - 81\sqrt{t(c+t)}) \geq 0, \quad (41) \\ & \quad \text{if and only if } c \leq \frac{3(1487 - 27\sqrt{2761})t}{200} \sim 1.024t, \\ \phi_2'\left(-1 + \sqrt{4-\frac{c}{t}}\right) &= 4(2t - \sqrt{t(4t-c)})(27\sqrt{t(4t-c)} - 8c - 30t) \geq 0, \quad (42) \\ & \quad \text{if and only if } c \leq \frac{3(9\sqrt{2713} - 403)t}{128} \sim 1.542t. \end{aligned}$$

First, we assume that $c \leq \frac{3(1487-27\sqrt{2761})t}{200} \sim 1.024t$. Then $\phi_2' \geq 0$ for this range. Thus, the right corner of this range maximizes $\phi_2$. Substituting $l_1 = 2 - \sqrt{1+c/t}$ and $l_1 = -1 + \sqrt{4-c/t}$ (the corners of the range) into $\phi_2(l_1)$, we have:

$$\begin{aligned} \phi_2\left(2 - \sqrt{1+\frac{c}{t}}\right) &= 4(31c + 126t)\sqrt{t(c+t)} - 2(5c^2 + 188ct + 252t^2) \leq 0, \quad (43) \\ & \quad \text{if and only if } c \leq \frac{2(491 - 26\sqrt{331})t}{25} \sim 1.438t, \\ \phi_2\left(-1 + \sqrt{4-\frac{c}{t}}\right) &= 2((16t - 3c)(36t + c) - 2(144t - 5c)\sqrt{t(4t-c)}) \leq 0, \quad (44) \\ & \quad \text{if and only if } c \leq \frac{2(11\sqrt{241} - 163)t}{9} \sim 1.726t. \end{aligned}$$

(43) and (44) imply that $\phi_2$ is negative since we assume that $c \leq \frac{3(1487-27\sqrt{2761})t}{200} \sim 1.024t (< 1.438t)$. Thus, the expected profit is decreasing in $l_1$. Therefore, $l_1 = 2 - \sqrt{1+c/t}$ (the left corner of the range) is the best response of firm 1.

Second, we assume that $\frac{3(1487-27\sqrt{2761})t}{200} < c \leq \frac{2(491-26\sqrt{331})t}{25} \sim 1.438t$. Substituting $l_1 = -1 + \sqrt{4-c/t}$ (the right corner of the range) into $\phi_2''(l_1)$, we have:

$$\begin{aligned} \phi_2''\left(-1 + \sqrt{4-\frac{c}{t}}\right) &= 8t(13c + 6t - 9\sqrt{t(4t-c)}) > 0, \quad (45) \\ & \quad \text{if and only if } c > \frac{3(3\sqrt{3097} - 79)t}{338} \sim 0.781t. \end{aligned}$$

Since $\phi_2''$ is decreasing, it implies that $\phi_2''$ is positive for this range. Note that we assume $c > \frac{3(1487-27\sqrt{2761})t}{200} \sim 1.024t$. Since $\phi_2$ is convex, either of the two corners maximizes $\phi_2$. (43) and

(44) imply that $\phi_2$ is negative for the range (ii-2). Thus, the expected profit is decreasing in $l_1$. Therefore, $l_1 = 2 - \sqrt{1 + c/t}$ (the left corner of the range) is the best response of firm 1.

Third, we assume that $\frac{2(491-26\sqrt{331})t}{25} \sim 1.438t < c \leq \frac{2(11\sqrt{241}-163)t}{9} \sim 1.726t$. As we show in the previous paragraph, $\phi_2$ is convex. Since $\phi_2$ is positive at the left corner of the range (ii-2) and it is negative at the right corner of the range (ii-2), there is an $l_1$ satisfying $\phi_2 = 0$, and it maximizes the expected profit of firm 1. Let the above maximizer be $\bar{l_1}$. For this range (ii-2) $l_1 = \bar{l_1}$ is firm 1's best reply.

Fourth, we assume that $\frac{2(11\sqrt{241}-163)t}{9} \sim 1.726t \leq c$. Since $\phi_2'' > 0$, $\phi_2'$ is increasing. Since $\phi_2'$ is non-positive at the right corner, $\phi_2'$ is always non-positive. Thus, the right corner minimizes $\phi_2$. (44) implies that $\phi_2$ is always non-negative, so the expected profit of firm 1 is non-decreasing. Therefore, $l_1 = -1 + \sqrt{4 - c/t}$ (the right corner) is the best response of firm 1.

We summarize the best response of firm 1 for the range (ii-2). If $c \leq \frac{2(491-26\sqrt{331})t}{25}$, the best response is $l_1 = 2 - \sqrt{1 + c/t}$; if $\frac{2(491-26\sqrt{331})t}{25} < c < \frac{2(11\sqrt{241}-163)t}{9}$, it is $\bar{l_1}$; and if $\frac{2(11\sqrt{241}-163)t}{9} \leq c$, it is $l_1 = -1 + \sqrt{4 - c/t}$. **Q.E.D.**

**Lemma A3:** *Suppose that $l_2 = 1$. Suppose that $c < 3t$. The expected profit of firm 1 is non-increasing in $l_1$ for all $l_1$ belonging to the range (iii-2) if $c \leq \sqrt{6}t \sim 2.449t$.*

**Proof:** Since $c < 3t$ and $0 < 2 - \sqrt{1 + \frac{c}{t}}$, the range (iii-2) is never empty. From (29), we have that $D_1 < 1$ when $c_1 < c_2$. From (30), we have that $D_1 > 0$ when $c_2 < c_1$. Thus, the expected profit of firm 1 is given by

$$\begin{aligned} E[\Pi_1(l_1, l_2)] \; = \; & \frac{1}{4} \times \frac{t(l_2 - l_1)(2 + l_1 + l_2)^2}{18} + \frac{1}{4} \times \frac{t(l_2 - l_1)(2 + l_1 + l_2)^2}{18} \\ & + \frac{1}{4} \times \frac{(c + t(l_2 - l_1)(2 + l_1 + l_2))^2}{18t(l_2 - l_1)} + \frac{1}{4} \times \frac{(t(l_2 - l_1)(2 + l_1 + l_2) - c)^2}{18t(l_2 - l_1)}. \end{aligned} \tag{46}$$

The first and the second terms are the expected profits in which both firms' costs are the same. The third term is the expected profit in which $c_1 = 0$ and $c_2 = c$. The fourth term is the expected profit in which $c_1 = c$ and $c_2 = 0$. Substituting $l_2 = 1$ into $E[\Pi_1(l_1, l_2)]$ in (46) and differentiating it with respect to $l_1$, we have

$$\frac{\partial E[\Pi_1(l_1, 1)]}{\partial l_1} = \frac{\psi_2(l_1)}{36t(1 - l_1)^2}, \tag{47}$$

where $\psi_2(l_1) \equiv c^2 - 2t^2(1-l_1)^2(3+l_1)(1+3l_1)$. We now show that $\psi_2(l_1) \leq 0$ for all $l_1$ belonging to the range (iii-2). Differentiating $\psi_2(l_1)$, we have

$$\psi_2'(l_1) = 8t^2(1-l_1)(-1+6l_1+3l_1^2) < 0, \text{ if and only if } l_1 < \frac{2\sqrt{3}-3}{3}. \tag{48}$$

(48) implies that either of the two corners ($l_1 = 0$ or $l_1 = 2 - \sqrt{1+c/t}$) maximizes $\psi_2$. $\psi_2(0) \leq 0$ if $c \leq \sqrt{6}t \sim 2.449t$. Substituting $l_1 = 2 - \sqrt{1+c/t}$ into $\psi_2(l_1)$, we have

$$\begin{aligned} \psi_2\left(2 - \sqrt{1+\frac{c}{t}}\right) &= -(5c^2 + 176ct + 240t^2) + 8(7c+30t)\sqrt{t(t+c)} \leq 0, & (49) \\ & \text{if and only if } c \leq \frac{4(172 - 23\sqrt{46})t}{25} \sim 2.561t. & (50) \end{aligned}$$

Thus, $\psi_2(l_1) \leq 0$ if $c \leq \sqrt{6}$. **Q.E.D.**

We now consider the optimal location of firm 1 for the whole range, i.e., [0,1].

**Lemma A4:** *Suppose that $l_2 = 1$. Suppose that $c < 3t$. Suppose that $c > \frac{t}{18}\left(81 - \sqrt{3(1928\sqrt{241} - 29269)}\right) \sim 2.025t$. Then $l_1 = 0$ never maximizes the expected profit of firm 1.*

**Proof:** From Lemmas A1–A3, we have that the expected profit of firm 1 is locally maximized when $l_1 = \frac{\sqrt{241}-14}{3}$. If $E[\Pi_1(\frac{\sqrt{241}-14}{3}, 1)] > E[\Pi_1(0,1)]$, $l_1 = 0$ does not maximize the expected profit of firm 1. The condition is

$$\begin{aligned} & \frac{(482\sqrt{241} - 7378)t + 243c}{972} - \frac{c^2 + 18t^2}{36t} > 0 \\ \Leftrightarrow \quad & c > \frac{t}{18}\left(81 - \sqrt{3(1928\sqrt{241} - 29269)}\right) \sim 2.025t. \end{aligned} \tag{51}$$

**Q.E.D.**

**Lemma A5:** *Suppose that $l_2 = 1$. Suppose that $c < 3t$. Suppose that $c \leq \frac{t}{18}\left(81 - \sqrt{3(1928\sqrt{241} - 29269)}\right) \sim 2.025t$. Then $l_1 = 0$ maximizes the expected profit of firm 1.*

**Proof:** Suppose that $c \leq \frac{2(491 - 26\sqrt{331})t}{25} \sim 1.438t$. From Lemmas A1–A3 we have that the expected profit of firm 1 is non-increasing in $l_1$ for all $l_1 \in [0,1]$, so the maximizer is $l_1 = 0$. Note that the expected profit is continuous with respect to $l_1$.

Suppose that $\frac{t}{18}\left(81-\sqrt{3(1928\sqrt{241}-29269)}\right) \sim 2.025t \geq c \geq \frac{2(11\sqrt{241}-163)t}{9} \sim 1.726t$. From Lemmas A1–A3 we have that the expected profit of firm 1 is locally maximized when $l_1 = 0$ and when $l_1 = \frac{\sqrt{241}-14}{3}$. Note that the expected profit is continuous with respect to $l_1$. From (51), we have that the profit is larger when $l_1 = 0$ than when $l_1 = \frac{\sqrt{241}-14}{3}$.

Suppose that $\frac{2(11\sqrt{241}-163)t}{9} \sim 1.726t > c > \frac{2(491-26\sqrt{331})t}{25} \sim 1.438t$. From Lemmas A1–A3 we have that the expected profit of firm 1 is locally maximized when $l_1 = 0$ and when $l_1 = \bar{l_1}$, where $\bar{l_1}$ is defined in the proof of Lemma A2. We show that the expected profit is larger when $l_1 = 0$ than when $l_1 = \bar{l_1}$.

From $E[\Pi_1(l_1, 1)]$ in (36), we have the second, the third, and the fourth derivatives:

$$\frac{\partial^2 E[\Pi_1(l_1,1)]}{\partial l_1^2} = \frac{c^2 - 3t^2(1-l_1)^3(11+3l_1)}{36t(1-l_1)^3}, \tag{52}$$

$$\frac{\partial^3 E[\Pi_1(l_1,1)]}{\partial l_1^3} = \frac{c^2 - 3t^2(1-l_1)^4}{12t(1-l_1)^4}, \tag{53}$$

$$\frac{\partial^4 E[\Pi_1(l_1,1)]}{\partial l_1^4} = \frac{c^2}{3t(1-l_1)^5} > 0. \tag{54}$$

(54) implies that (53) is increasing in $l_1$. We show that (53) is positive for all $l_1$ belonging to the range (ii-2). Substituting $l_1 = 2 - \sqrt{1 + c/t}$ (the left corner of range (ii-2)) into (53), we have

$$\left.\frac{\partial^3 E[\Pi_1(l_1,1)]}{\partial l_1^3}\right|_{l_1 = 2-\sqrt{1+\frac{c}{t}}} = \frac{t^3\left(6(2t+c)\sqrt{t(t+c)} - (12t^2+12ct+c^2)\right)}{6(\sqrt{t(c+t)}-t)^4}. \tag{55}$$

Since $(6(2t+c)\sqrt{t(t+c)})^2 - ((12t^2+12ct+c^2))^2 = c^2(12t^2+12ct-c^2) > 0$, (55) is positive. Thus (55) is positive for any $l_1$ lying on ragne (ii-2). Since (53) is positive, (52) is increasing in $l_1$. We show that (52) is negative for all $l_1$ belonging to the range (ii-2). Substituting $l_1 = -1 + \sqrt{4 - c/t}$ (the right corner of range (ii-2)) into the numerator of (52), we have

$$\begin{aligned} & c^2 - 3t^2\left(1 - \frac{-t+\sqrt{t(4t-c)}}{t}\right)^3\left(11 + 3\frac{-t+\sqrt{t(4t-c)}}{t}\right) \\ = \;& 2(3(16t+5c)\sqrt{t(4t-c)} - (96t^2+18ct-5c^2)). \end{aligned}$$

Since $\left(3(16t+5c)\sqrt{t(4t-c)}\right)^2 - \left(96t^2+18ct-5c^2\right)^2 = -c^2(25c^2+45ct-96t^2) < 0$, (52) is negative. Since the expected profit of firm 1 is concave, the following inequality holds:

$$\begin{aligned}
E[\Pi_1(\bar{l}_1,1)] &= E\left[\Pi_1\left(2-\sqrt{1+\frac{c}{t}},1\right)\right] + \int_{2-\sqrt{1+c/t}}^{\bar{l}_1} \frac{\partial E[\Pi_1(l_1,1)]}{\partial l_1} dl_1 \\
&< E\left[\Pi_1\left(2-\sqrt{1+\frac{c}{t}},1\right)\right] \\
&\quad + \left(-1+\sqrt{4-\frac{c}{t}}-\left(2-\sqrt{1+\frac{c}{t}}\right)\right) \frac{\partial E[\Pi_1(l_1,1)]}{\partial l_1}\bigg|_{l_1=2-\sqrt{1+\frac{c}{t}}} \equiv H.
\end{aligned}$$

If $H < E[\Pi_1(0,1)]$, for any $c \in [1.438t, 1.726t]$, $l_1 = \bar{l}_1$ is not the best response of firm 1.

$$\begin{aligned}
E[\Pi_1(0,1)] - H &= \frac{c^2+18t^2}{36t} - \left(E\left[\Pi_1\left(2-\sqrt{1+\frac{c}{t}},1\right)\right]\right. \\
&\quad \left. + \left(-1+\sqrt{4-\frac{c}{t}}-\left(2-\sqrt{1+\frac{c}{t}}\right)\right) \frac{\partial E[\Pi_1(2-\sqrt{1+\frac{c}{t}},1)]}{\partial l_1}\right) \\
&= \frac{2t^3(342t+127c)\sqrt{t(c+t)} + t^2(252t^2+188tc+5c^2)\sqrt{t(4t-c)}}{36t^2(\sqrt{t(c+t)}-t)^2} \\
&\quad - \frac{t^2(684t^3+596t^2c+48tc^2-c^3)+2t^2(126t+31c)\sqrt{t(c+t)}\sqrt{t(4t-c)}}{36t^2(\sqrt{t(c+t)}-t)^2}, \\
&= \frac{H_a - H_b}{36t^2(\sqrt{t(c+t)}-t)^2}, \\
\text{where} \quad H_a &\equiv 2t^3(342t+127c)\sqrt{t(c+t)} + t^2(252t^2+188tc+5c^2)\sqrt{t(4t-c)}, \\
H_b &\equiv t^2(684t^3+596t^2c+48tc^2-c^3) + 2t^2(126t+31c)\sqrt{t(c+t)}\sqrt{t(4t-c)}.
\end{aligned}$$

$H_a > 0$ and $H_b > 0$. We square $H_a$ and $H_b$:

$$\begin{aligned}
H_a^2 &= t^5(721872t^5+1130832t^4c+468692t^3c^2+34172t^2c^3-1780tc^4-25c^5) \\
&\quad +4t^5(342t+127c)(252t^2+188tc+5c^2)\sqrt{t(t+c)}\sqrt{t(4t-c)}, \qquad (56) \\
H_b^2 &= t^4(721872t^6+1130832t^5c+466496t^4c^2+36132t^3c^3-2732t^2c^4-96tc^5+c^6) \\
&\quad +4t^4(126t+31c)(684t^3+596t^2c+48tc^2-c^3)\sqrt{t(t+c)}\sqrt{t(4t-c)}. \qquad (57)
\end{aligned}$$

If $H_a^2 - H_b^2 > 0$, then $E[\Pi_1(0,1)] - H > 0$.

$$\begin{aligned}
H_a^2 - H_b^2 &= t^4c^2(2196t^4-1960t^3c+952t^2c^2+71tc^3-c^4) \\
&\quad -4t^4c^2(-1062t^2+727tc-31c^2)\sqrt{t(t+c)}\sqrt{t(4t-c)} = H_c - H_d,
\end{aligned}$$

$$
\begin{aligned}
\text{where}\quad H_c \equiv\ & t^4c^2(2196t^4 - 1960t^3c + 952t^2c^2 + 71tc^3 - c^4),\\
H_d \equiv\ & 4t^4c^2(-1062t^2 + 727tc - 31c^2)\sqrt{t(t+c)}\sqrt{t(4t-c)}.
\end{aligned}
$$

$H_c > 0$ and $H_d > 0$ (note that, $c \in [1.438t, 1.726t]$). If $H_c^2 - H_d^2 > 0$, then $H_a^2 - H_b^2 > 0$, so $E[\Pi_1(0,1)] - H > 0$. We now show $H_c^2 - H_d^2 > 0$. $H_c^2 - H_d^2 = t^8c^4(3t-c)(g(c))$, where

$$
\begin{aligned}
g(c) \quad \equiv \quad & -22453200t^7 + 4542480t^6c + 22230000t^5c^2 - 10513848t^4c^3\\
& +573920t^3c^4 - 18096t^2c^5 + 139tc^6 - c^7.
\end{aligned}
$$

Differentiating $g(c)$, we have (note that, $c \in [1.438t, 1.726t]$)

$$
\begin{aligned}
g'(c) &= 4542480t^6 + 44460000t^5c - 31541544t^4c^2 + 2295680t^3c^3 - 90480t^2c^4 + 834tc^5 - 7c^6,\\
g''(c) &= 6(7410000t^5 - 10513848t^4c + 1147840t^3c^2 - 60320t^2c^3 + 695tc^4 - 7c^5),\\
g^{(3)}(c) &= -6(10513848t^4 - 2295680t^3c + 180960t^2c^2 - 2780tc^3 + 35c^4),\\
g^{(4)}(c) &= 120(114784t^3 - 18096t^2c + 417tc^2 - 7c^3),\\
g^{(5)}(c) &= -360(6032t^2 - 278tc + 7c^2),\\
g^{(6)}(c) &= 720(139t - 7c) > 0.
\end{aligned}
$$

From the last inequality, we have that $g^{(5)}(c)$ is increasing. Substituting the right corner ($c = 1.726t$) into $g^{(5)}(c)$ yields $g^{(5)}(c) < 0$, so $g^{(4)}(c)$ is decreasing. Substituting the left corner ($c = 1.438t$) into $g^{(4)}(c)$ yields $g^{(4)}(c) > 0$, so $g^{(3)}(c)$ is increasing. Substituting the right corner into $g^{(3)}(c)$ yields $g^{(3)}(c) < 0$, so $g^{(2)}(c)$ is decreasing. Substituting the right corner into $g''(c)$ yields $g^{(2)}(c) < 0$. Since $g$ is concave, either of the two corners minimizes $g$. Substituting both corners into $g(c)$, we have that $g(c) > 0$. **Q.E.D.**

Next, we consider the cases in which $c > 3t$.

**Lemma A6:** *Suppose that $l_2 = 1$. Suppose that $c \geq 3t$. $l_1 = 0$ never maximizes the expected profit of firm 1.*

**Proof:** As we have already showed, both (29) and (30) are satisfied. The expected profit is always given by (33). Thus, (51) holds true for $c \geq 3t$, too. (51) implies that Lemma A6 holds. **Q.E.D.**

Finally we prove Proposition 2.

Lemma A4 and A6 imply the only if part of Proposition 2. Lemma A5 implies the if part of Proposition 2. **Q.E.D.**

**Proof of Proposition 4:** In the model, there are four possible patterns of cost allocations: (i) $c_1 = c_2 = 0$, (ii) $c_1 = c_2 = c$, (iii) $c_1 = 0$ and $c_2 = c$, and (iv) $c_1 = c$ and $c_2 = 0$.

When each firm locates at $l_1 = l_2 = 1/2$, the most efficient firm supplies for all consumers in any case. The transportation costs of consumers are

$$\int_0^{\frac{1}{2}} t(1/2 - x)^2 dx + \int_{\frac{1}{2}}^1 t(x - 1/2)^2 dx = \frac{t}{12}.$$

When each firm locates at $l_1 = 0$ and $l_2 = 1$, the less efficient firm may supply for some consumers in cases (iii) and (iv). The quantity supplied by the more efficient firm is larger than 1/2. The transportation costs of consumers are

$$\int_0^{D_1} tx^2 dx + \int_{D_1}^1 t(1-x)^2 dx = \frac{t(1 - 3D_1 + 3D_1^2)}{3} = \frac{t(1/4 + (D_1 - 1/2)^2)}{3} > \frac{t}{12}.$$

In any case, total surplus in the minimum differentiation is larger than or equal to that in the maximum differentiation. Therefore, Proposition 4 holds. **Q.E.D.**

**Proof of Proposition 5:** We first consider the profits in which both firms locate at $l_1 = l_2 = 1/2$. In the case, unless a firm has the cost advantage, the profit of the firm is zero. If the firm has the cost advantage, the profit is $c$. From (20), the expected profit is

$$E[\pi_1(1/2, 1/2)] = E[\pi_2(1/2, 1/2)] = \frac{c}{4}. \tag{58}$$

We consider the profits in which each firm locates at $l_1 = 0$ and $l_2 = 1$. If $c < 3t$, each firm has a positive profit in any case. From (46), the expected profit is

$$E[\Pi_1(0,1)] = E[\Pi_2(0,1)] = \frac{1}{4} \times \frac{t}{2} + \frac{1}{4} \times \frac{t}{2} + \frac{1}{4} \times \frac{(c+3t)^2}{18t} + \frac{1}{4} \times \frac{(3t-c)^2}{18t} = \frac{c^2 + 18t^2}{36t}. \tag{59}$$

Since (58) − (59) $= (3t - c)(6t - c)/36t > 0$ as long as $c < 3t$, the former part of Proposition 5 holds. When $c \geq 3t$, a firm does not have a positive profit if it has the cost disadvantage. From

(33), the expected profit is

$$E[\Pi_1(0,1)] = E[\Pi_2(0,1)] \quad = \quad \frac{1}{4} \times \frac{t}{2} + \frac{1}{4} \times \frac{t}{2} + \frac{1}{4} \times (c - t) = \frac{c}{4}. \tag{60}$$

(59) and (60) imply the latter part of Proposition 5. **Q.E.D.**

## References


Bester, H. "Quality Uncertainty Mitigates Product Differentiation." *RAND Journal of Economics*, Vol. 29 (1998), pp. 828–844.

Boyer, M., Laffont, J.J. Mahenc, P. and Moreaux, M. "Location Distortions under Incomplete Information." *Regional Science and Urban Economics*, Vol. 24 (1994), pp. 409–440.

Boyer, M., Mahenc, P. and Moreaux, M. "Asymmetric Information and Product Differentiation." *Regional Science and Urban Economics*, Vol. 33 (2003), pp. 93–113.

Cardon, J. H. and Sasaki, D. "Preemptive Ssearch and R&D Clustering." *RAND Journal of Economics*, Vol. 29 (1998), pp. 324–338.

d'Aspremont, C., Gabszewicz, J. J. and Thisse, J.-F. "On Hotelling's Stability in Competition." *Econometrica*, Vol. 47 (1979), pp. 1145–1150.

De Palma, A., Ginsburgh, V., Papageorgiou, Y. Y. and Thisse, J. -F. "The Principle of Minimum Differentiation Holds under Sufficient Heterogeneity." *Econometrica*, Vol. 53 (1985), pp. 767–781.

Friedman, J. W. and Thisse, J. -F. "Partial Collusion Fosters Minimum Product Differentiation." *Rand Journal of Economics*, Vol. 24 (1993), pp. 631–645.

Hotelling, H. "Stability in Competition." *Economic Journal*, Vol. 39 (1929), pp. 41–57.

Ishibashi, K. "Strategic Delegation under Quality Competition." *Journal of Economics*, Vol. 73 (2001), pp. 25–56.

Lahiri, S. and Ono, Y. "Helping Minor Firms Reduces Welfare." *Economic Journal*, Vol. 98 (1988), pp. 1199–1202.

Lahiri, S. and Ono, Y. "Foreign Direct Investment, Local Content Requirement, and Profit Taxation." *Economic Journal*, Vol. 108 (1998), pp. 444–457.

Ma, C. A. and Burgess, J. F. "Quality Ccompetition, Welfare, and Regulation." *Journal of Economics*, Vol. 58 (1993), pp. 153–173.

Mai, C. C. and Peng, S. K. "Cooperation vs. Competition in a Spatial Model." *Regional Science and Urban Economics*, Vol. 29 (1999), pp. 463–472.

Matsumura, T. "Partial Privatization in Mixed Duopoly." *Journal of Public Economics*, Vol. 70 (1998), pp. 473–483.

Ono, Y. "Foreign Penetration and National Welfare under Oligopoly." *Japan and the World Economy*, Vol. 2 (1990), pp. 141–154.

Riordan, M. H. "Anticompetitive Vertical Integration by a Dominant Firm." *American Economic Review*, Vol. 88 (1998), pp. 1232–1248.

Schulz, N. and Stahl, K. "On the Non-existence of Oligopolistic Equilibria in Differentiated Products Spaces." *Regional Science and Urban Economics*, Vol. 15 (1985), pp. 229–243.

Ziss, S. "Entry Deterrence, Cost Advantage and Horizontal Product Differentiation." *Regional Science and Urban Economics*, Vol. 23 (1993), pp. 523–543.